A Parameter Space Approach for Fixed-order robust controller synthesis by symbolic computation

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Abstract

In this paper we present a new parameter space design method for robust control synthesis, in particular in terms of real stability radius, using quantifier elimination (QE). We also aim at practicality by employing the scheme to combine sign definition condition (SDC) and special QE algorithm using Sturm-Habicht sequence. We show some concrete examples demonstrating the validity of our approach.

1 Introduction

For robust control synthesis and multi-objective design, a parameter space approach is known to be one of the effective methods. The parameter space approach can be utilized to determine the set of certain parameters which satisfies the given specifications in a parameter space. Recently, the parameter space design accomplished by using quantifier elimination (QE) has been proposed for robust multi-objective design problems [4, 6]: The robust control problems are reduced to first-order formula descriptions, then can be solved by applying general QE. However, *naive* reduction of the control problems to the QE problems, in general, complicated to achieve QE computation efficiently. This is a serious issue in view of efficiency because the worst-case complexity of general QE algorithm based on cylindrical algebraic decomposition (CAD) algorithm has doubly exponential behavior.

While, fortunately, many important design specifications for robustness can be reduced to the condition :

$$\forall x > 0 \ (f(x) > 0)$$

called *sign definite conditions (SDC)*. Moreover, for the SDC we can use a special QE algorithm using Sturm-Habicht sequence which is much more efficient than the general one. This scheme of combining reduction of the specifications to SDC and usage of special QE was first successfully introduced to solve robust control design problems in [1].

The robust controller synthesis problems, which are the problems of finding an appropriate fixed-order controller to achieve stability and a prescribed level of parameter stability margin (stability radius) for a plant, is as yet unsolved. Currently, in an engineering sense, mainly, the techniques for exact computation of stability radius can itself be used in an interactive loop to adjust target parameters to robustify the system. In this paper we propose a systematic approach to such robust controller synthesis problem using

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quantifier elimination and also aim at practicality. This is realized by utilizing the scheme for robust control design by [1].

The organization of the rest of the paper is as follows: The idea of robust control synthesis based on SDC and special QE algorithm is explained in §2. §3 is devoted to our QE approach to robust control analysis. §4 provides our QE approach to various synthesis problems. We show several concrete analysis and synthesis examples demonstrating the validity of our approach in §5. §6 addresses the concluding remarks.

2 Parametric approach to robust control design via QE

Consider a feedback control system shown in Fig 1. $\mathbf{p} = [p_1, p_2, \cdots, p_s]$ is the vector of uncertain real parameters in the plant G. $\mathbf{x} = [x_1, x_2, \cdots, x_t]$ is the vector of real parameters of controller C. Assume that the controller considered here is of fixed order.

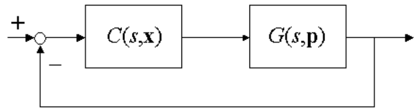

Figure 1: A standard feedback system

The performance of the control system can often be characterized by a vector $\mathbf{a} = [a_1, \cdots, a_l]$ which are functions of the plant and controller parameters $\mathbf{p}$ and $\mathbf{x}$:

$$a_i = a_i(\mathbf{x}, \mathbf{p}), \quad i = 1, \cdots, l. \tag{1}$$

H_∞-norm constraints, gain and phase margins and $\mathcal{D}$-stability condition *etc* are such performances, which are frequently used as the indices of robustness. Then the target specifications are usually given as follows for a desired value τ_i:

$$a_i(\mathbf{x}, \mathbf{p}) < \tau_i, \quad i = 1, \cdots, l. \tag{2}$$

Then the goal is to find the region in the parameter space which satisfy the specifications. Several concrete robust synthesis problems will be shown in §4.

For such problems QE based approach is one of the effective tools if the specifications (2) are reduced to first-order formula descriptions. Due to the inherent undesired computational complexity of general QE algorithm, aiming at practical applicability, it is important to employ the strategy to combine well the reduction of the specification conditions to "*simple* or *small*" QE problem and the efficient specialized algorithm for particular type of inputs. A successful example of such attempts is the scheme of combination of SDC and special QE algorithm using Sturm-Habicht sequence presented in [1]. The most specifications (such as H_∞-norm constraints, frequency restricted norms, phase/gain margins, $\mathcal{D}$-stability constraint) for robustness can result in sign definite conditions

$$\forall x > 0 \ (f(x, \mathbf{x}, \mathbf{p}) > 0). \tag{3}$$

and be dealt with in this scheme. Concrete comparison data of the efficiency to solve SDC by the general QE algorithm based on CAD and the special one using Sturm-Habicht sequence is shown in [1]. The results shows, in case of the moderate number of parameters, this scheme is quite effective for the rather high order system (more than ten-th order) that is considered to be practical in an engineering sense.

In this paper we focus on the real stability radius computation. Though there are many results about an explicit formula to compute the stability radius so far (*e.g.* [8, 5]), the problem of determining the fixed-order controller to achieve stability and a desired level of parameter stability margin is unsolved. Therefore here we extend the above scheme toward this robust controller synthesis problem.

3 Real Stability radius

In many control systems plant parameters may vary over a wide range on a nominal value $\mathbf{p}^0 = [p_1^0, \cdots, p_\ell^0]$. If the controller is given, the maximal range of variation of the parameter $\mathbf{p} = [p_1, \cdots, p_\ell]$, measured in a suitable norm, for which the stability is preserved, is the parametric stability margin (radius of stability) with the controller $\mathbf{x}$:

$$\rho_m = \sup\{\ r \mid g(s, \mathbf{x}, \mathbf{p})\ stable,\ ||\mathbf{p} - \mathbf{p}^0|| < r\},$$

where $g(s, \mathbf{x}, \mathbf{p})$ is the characteristic polynomial of the closed-loop system (shown in Fig 1).

We consider the following type of characteristic polynomial, *i.e.* its coefficients are linear function of the plant parameter $\mathbf{p}$:

$$g(s, \mathbf{x}, \mathbf{p}) = a_1(s, \mathbf{x})p_1 + \cdots + a_\ell(s, \mathbf{x})p_\ell + b(s, \mathbf{x}),$$

where $a_i(s)$ and $b(s)$ are polynomials over the reals $\mathbf{R}$ and $p_i \in bfR$. We refer to this as the *linear case*. We assume that $g(s, \mathbf{x}, \mathbf{p})$ has fixed degree.

3.1 An explicit formula

Here we employ the results from [2] and show their results briefly: Let

$$\Delta\mathbf{p} = \mathbf{p} - \mathbf{p}^0 = [\Delta p_1, \cdots, \Delta p_\ell].$$

We consider the linear case, so the characteristic polynomial of the system of Fig.1. can be written as follows:

$$g(s, \mathbf{p}^0 + \Delta\mathbf{p}) = g(s, \mathbf{p}^0) + \sum_{i=0}^{\ell} a_i(s)\Delta p_i. \tag{4}$$

Let s^* be a point on the stability boundary $\partial\mathcal{D}$. For s^* to be a root of $g(s, \mathbf{p}^0 + \Delta\mathbf{p})$, we have

$$g(s^*, \mathbf{p}^0) + \sum_{i=0}^{\ell} a_i(s^*)\Delta p_i = 0. \tag{5}$$

Taking account for weighted perturbations, we can rewrite above equation as follows:

$$g(s^*, \mathbf{p}^0) + \sum_{i=0}^{\ell} \frac{a_i(s^*)}{w_i} w_i \Delta p_i = 0. \tag{6}$$

where $w_i > 0$. The minimum norm solution of (6) gives us $\rho(s^*)$:

$$\rho(s^*) = \inf\{||\Delta\mathbf{p}|| \mid \Delta\mathbf{p}\ satisfies\ (6)\} \tag{7}$$

For (6), there are two cases: If s^* is real, (6) can be written as

$$\mathbf{A}(s^*)\mathbf{u}(s^*) = \mathbf{b}(s^*), \tag{8}$$

where

$$\mathbf{A}(s^*) = \begin{bmatrix} \frac{a_1(s^*)}{w_1} & \cdots & \frac{a_\ell(s^*)}{w_\ell} \end{bmatrix},$$
$$\mathbf{u}(s^*) = [w_1\Delta p_1, \cdots, w_\ell \Delta p_\ell]^T,$$
$$\mathbf{b}(s^*) = -g(s^*, \mathbf{p}^0).$$

If s^* is complex, (6) can be written as the same equation (8) with

$$\mathbf{A}(s^*) = \begin{bmatrix} \frac{a_{R,1}(s^*)}{w_1} & \cdots & \frac{a_{R,\ell}(s^*)}{w_\ell} \\ \frac{a_{I,1}(s^*)}{w_1} & \cdots & \frac{a_{I,\ell}(s^*)}{w_\ell} \end{bmatrix},$$
$$\mathbf{u}(s^*) = [w_1\Delta p_1, \cdots, w_\ell \Delta p_\ell]^T,$$
$$\mathbf{b}(s^*) = [-g_R^0, -g_I^0]^T,$$
$$a_{R,k} = Re(a_k(s^*)),\ a_{I,k} = Im(a_k(s^*)),$$
$$g_R^0 = Re(g(s^*, \mathbf{p}^0)),\ g_I^0 = Im(g(s^*, \mathbf{p}^0)).$$

The equation (8) determines the parametric stability margin in any norm: Let u^* be the minimum norm solution of (8). Then

$$\rho = ||u^*||.$$

If (8) has no solution, then ρ is set equal to ∞. Now consider ℓ^2-norm case. Assume that $\mathbf{A}$ has full rank $= 2$, then the minimum norm solution u^* is given as follows:

$$u^*(s) = \mathbf{A}^T(s)[\mathbf{A}(s)\mathbf{A}^T(s)]^{-1}\mathbf{b}(s). \tag{9}$$

3.2 QE approach to robust control analysis

First we consider the case where s^* is real. Let a finite set of intersection points between $\partial\mathcal{D}$ and real axis be $\{r_1, \cdots, r_k\}$. For each r_i we can compute the $\rho(r_i) \equiv ||u^*(r_i)||_2$ from (9) immediately.

As for the case where s^* is complex we use an appropriate parametrization $\alpha(t)$ of the stability domain boundary $\partial\mathcal{D}$, where $t \in I = [t_s, t_e] \subset \mathbf{R} \cup \{\pm\infty\}$. We allow only polynomial descriptions for the parametrization. (*e.g.* Hurwitz case $\alpha(t) = \mathbf{i}t$ where $\mathbf{i}$ is an imaginary unit). We substitute the parametrization $\alpha(t)$ for the indeterminate s in the formula (9), resulting in an expression $u^*(\alpha(t))$. We simply denote $u^*(\alpha(t))$ by $u^*(t)$.

Having an explicit formula $u^*(t)$ enables us to compute the *exact* minimum of $||u^*(t)||_2$ with respect to $t \in I$ symbolically. Actually we compute the minimum $\mathcal{F}_m$ of

$$\mathcal{F}(t) = ||u^*(t)||_2^2. \tag{10}$$

Our main tool is a quantifier elimination.

Moreover, we must deal with the case $\mathbf{A}$ has less than full rank. If $rank(\mathbf{A}) = 0$, (8) has no solution, so $\rho = \infty$. If $rank(\mathbf{A}) = 1$, (8) is consistent iff $rank[\mathbf{A}, \mathbf{b}] = 1$, otherwise (8) has no solution, hence

$\rho = \infty$. Hence, for the case of $rank[\mathbf{A}, \mathbf{b}] = 1$, we simply replace two equations with a single equation and can proceed as before. Let $\{d_1, \cdots, d_m\}$ be the values of t for which the rank drops and $rank[\mathbf{A}, \mathbf{b}] = 1$.

Then the stability radius given by the following:

$$\rho_m = \min\{\sqrt{\mathcal{F}_m},\ \rho(r_i),\ \rho(\mathbf{i}d_j)\}. \tag{11}$$

Now consider to compute the minimum $\mathcal{F}_m$ of $\mathcal{F}(t)$. In general, $\mathcal{F}(t)$ is a rational polynomial, say $\mathcal{F}(t) = N(t)/D(t)$ for polynomials N, D. Finding the minimum of $\mathcal{F}(t)$, this type of optimization is called *hyperbolic optimization*, can be solved as the following QE problem:

$$\exists t \in I\ ((D > 0 \wedge N \leq zD) \vee (D < 0 \wedge N \geq zD))$$

where z is newly introduced variable corresponding to $\mathcal{F}(t)$. The denominator of $\mathcal{F}(t)$ is strictly positive. hence $D > 0$ is true, so the above formula can be reduced further to

$$\exists t \in I\ (N \leq zD). \tag{12}$$

By performing QE for (12) we have a equivalent quantifier-free formula $\Psi(z)$ which presents the possible range of z and, in particular, stating the minimal value of $\mathcal{F}(t)$. Equivalently, we can solve (12) by solving the following QE problem

$$\forall t \in I\ (N - zD > 0). \tag{13}$$

Performing QE for (13) gives a quantifier-free formula $\Phi(z)$ equivalent to (13). $\Phi(z)$ presents the possible range of z, and, in particular, stating the maximum value of z which corresponds to the minimum of $\mathcal{F}(t)$. In other words, $\Phi(z) = \neg\Psi(z)$.

The first-order formula of the type (13) can be reduced to the following SDC:

$$\forall y > 0\ (h(y) > 0) \tag{14}$$

by a bilinear transformation $y = -\frac{(t-t_s)}{(t-t_e)}$, where $h(y)$ is a polynomial. A special QE algorithm using Sturm-Habicht sequence introduced in [1] can be utilized for the SDC. This is why we employ the reduction (13) instead of (12). Let the resulting formula after applying QE to (14) be $\Pi(z)$. Then $\neg\Pi(z)$ shows the possible range of z stating the minimum of $\mathcal{F}(t)$.

4 Synthesis problems

In case of synthesis problems, the control parameters $\mathbf{x}$ remains as free parameters during the procedures in the previous section. Here we illustrate how we solve several concrete synthesis problems. First we consider the following basic problem:

Problem 1

Consider the control system in Fig.1. Given a specific value of stability radius ρ. Let $g(s, \mathbf{x}, \mathbf{p})$ be a characteristic polynomial of the closed-loop system with fixed degree and $\mathbf{p}^0$ be a vector of nominal values of plant parameters such that $g(s, \mathbf{x}, \mathbf{p}^0)$ is $\mathcal{D}$-stable. Then find feasible region of control parameters $\mathbf{x}$ to achieve the desired level ρ of stability radius.

$\mathcal{D}$-stability condition: First, we should mention the condition of parameters $\mathbf{x}$ so that $g(s, \mathbf{x}, \mathbf{p}^0)$ is $\mathcal{D}$-stable. The condition would be given as a semialgebraic set. For Hurwitz stability, such condition of $\mathbf{x}$ is given by the well-known Liénard-Chipart criterion immediately. For Schur stability and wedge shape regions, *i.e.* the domain $\mathcal{D}$ of which the complementary set $\overline{\mathcal{D}} = \mathbf{C} - \mathcal{D}$ is of the form $\overline{\mathcal{D}} = \{x(\omega, t) + \mathbf{i}y(\omega, t) \in \mathbf{C} | \omega \in \mathbf{R}, t \in [t_s, t_e]\}$, the pole location problem can also be reduced to check a sign definite condition, see [7]. So $\mathcal{D}$-stability condition is also solved efficiently by a special QE using Sturm-Habicht sequence.

Algorithm: Problem 1 is solved by the same procedure shown in §3.2: For the case where $s^* = r_i$ $(i = 1, \cdots, k)$ is real, immediately from (9), we have $\rho(r_i) = ||u^*(r_i)||_2$ as formulas in $\mathbf{x}$. Let $\delta = \rho^2$. We set

$$\psi_i(\mathbf{x}) \equiv (||u^*(r_i)||_2^2 \geq \delta), \quad for\ i = 1, \cdots, k$$

In the case where s^* is complex, the formula (13) is of the same form containing the parameters $\mathbf{x}$:

$$\forall t \in I\ (N(\mathbf{x}, t) - \delta \cdot D(\mathbf{x}, t) > 0). \tag{15}$$

Consequently, we lead to the following SDC;

$$\forall y > 0\ (h_p(\mathbf{x}, y) > 0), \tag{16}$$

where h_p is a polynomial. After performing QE for (16) we have an equivalent quantifier-free formula $\phi(\mathbf{x})$ showing possible range of $\mathbf{x}$ which satisfies the given stability radius condition. $\phi(\mathbf{x})$ is also a semialgebraic set in $\mathbf{x}$. Moreover, for the non full rank case, from (9), we have

$$\kappa(\mathbf{x}) \equiv (||u^*(\mathbf{i}d_j)||_2^2 \geq \delta) \quad for\ j = 1, \cdots, m$$

Finally, the formula

$$\Gamma(\mathbf{x}) \equiv\ \phi(\mathbf{x})\ \vee\ (\bigvee_i \psi_i(\mathbf{x}))\ \vee (\bigvee_j \kappa_j(\mathbf{x}))$$

gives possible region of $\mathbf{x}$ which satisfies the given stability radius specification.

Next we show some advanced synthesis problems including multi-objective problems and optimization which can be solved naturally by using our parameter space approach based on QE presented in this paper and [1].

Problem 2

Find the maximum attainable stability radius ρ by a fixed-order $\mathcal{D}$-stable controller $C(s, \mathbf{x})$.

Problem 3

Find the best achievable nominal performance by a fixed-order $\mathcal{D}$-stable controller under a stability radius constraint.

Problem 2: The attainable stability radius can be obtained by the same procedure as before if we leave ρ as a free parameter, resulting in the semialgebraic expression $\Gamma'(\rho, \mathbf{x})$. Moreover, the stability condition, say $\mathcal{S}(\mathbf{x})$, of the controller $C(s, \mathbf{x})$ is obtained by the same way shown above. Then the attainable stability radius condition is given by

$$\varphi_2(\rho, \mathbf{x}) \equiv \Gamma'(\rho, \mathbf{x}) \wedge \mathcal{S}(\mathbf{x}).$$

The maximum attainable stability radius is obtained by solving the optimization problem:

$$Maximize\ \rho \ \ subject\ to\ \varphi_2(\rho, \mathbf{x}). \tag{17}$$

Problem 3: Here consider the system with a given level of stability radius ρ_0. Let $T_i(s, \mathbf{x}, \mathbf{p}^0)$ be the transfer functions and

$$||T_i(s, \mathbf{x}, \mathbf{p}^0)||_{[\underline{\omega_i}, \overline{\omega_i}]} < \gamma_i \tag{18}$$

be the nominal performance specifications with frequency restrictions. Here $\gamma_i > 0$ are free parameters. The frequency restricted norm constraints (18) can be reduced to SDC and solved by our approach, resulting in the formulas $\mathcal{N}_i(\gamma_i, \mathbf{x})$, respectively. Therefore, the achievable nominal performances condition are given as

$$\varphi_3(\gamma_i, \mathbf{x}) \equiv \Gamma'(\rho_0, \mathbf{x}) \wedge \mathcal{S}(\mathbf{x}) \wedge \bigwedge_i \mathcal{N}_i(\gamma_i, \mathbf{x}).$$

The maximum achievable nominal performance obtained by solving the following optimization problem:

$$Maximize\ \gamma_i \ \ subject\ to\ \varphi_3(\gamma_i, \mathbf{x}). \tag{19}$$

Optimization (17)(19): In general, both the optimization problems are nonlinear and non-convex. Since φ_2, φ_3 are polynomial constraints, they can be solved by using QE exactly. However, we have to use general QE algorithm because generally the reduced QE problems are considered not to have a specific structure desirable in the computation. Hence, this is practical only for modest size of problem. Methods of numerical optimizations could be utilized for large size problems.

5 Analysis and Synthesis Examples

This section provides analysis and synthesis problems to confirm the validity of our approach[1)].

5.1 Stability radius computation

Consider the continuous time control system with the plant, from [2],

$$G(s, \mathbf{p}) = \frac{2s + 3 - \frac{1}{3}p_1 - \frac{5}{3}p_2}{s^3 + (4 - p_2)s^2 + (-2 - 2p_1)s + (-9 + \frac{5}{3}p_1 + \frac{16}{3}p_2))}$$

controlled by a PI controller

$$C(s) = 5 + \tfrac{3}{s}.$$

The characteristic polynomial $g(s, \mathbf{p})$ of the closed-loop system is as follows:

$$g(s, \mathbf{p}) = s^4 + (4 - p_2)s^3 + (8 - 2p_1)s^2 + (12 - 3p_2) + (9 - p_1 - 5p_2)$$

The nominal value of parameters are $\mathbf{p}^0 = [p_1^0, p_2^0] = [0, 0]$, for which the polynomial is stable. Now we compute ℓ^2-stability margin of this polynomial (with weights $w_1 = w_2 = 1$) in terms of Hurwitz sense,

[1)] Since QE problems in §5 are all not large the results can be obtained by using general QE package QEPCAD implemented by H.Hong *et. al.*, see [3]. The graphs of feasible regions are produced by using a computer algebra system Risa/Asir (cf. `http://www.math.kobe-u.ac.jp/Asir/asir.html`).

i.e., $\alpha(t) = \mathbf{i}t$ and $\{r_1 = 0\}$. For $r_1 = 0$, from (9), we have $\rho(0) = ||u^*(0)||_2 = \frac{9\sqrt{26}}{26}$. For $d_1 = \sqrt{3}$, (8) is consistent, and we have $\rho(\mathbf{i}\sqrt{3}) = \frac{3\sqrt{2}}{5}$. Then

$$\mathcal{F}(t) = ||u^*(t)||_2^2 = \frac{t^8 - 16t^6 - 22t^4 + 240t^2 + 105}{(2t^2 - 1)^2}.$$

The QE problem corresponding to (13) is

$$\forall t > 0(t^8 - 16t^6 + 106t^4 + 112t^2 + 137 - z(4t^4 - 4t^2 + 1) > 0).$$

By applying QE, we have $z - 16 < 0$. Since negation of this is $z - 16 \geq 0$, hence the minimum $\mathcal{F}(t) = 16$ and $\sqrt{\mathcal{F}_m} = 4$. Consequently, $\rho_m = \min(4, \frac{9\sqrt{26}}{26}, \frac{3\sqrt{2}}{5}) = \frac{3\sqrt{2}}{5}$. This result coincides with that in [2].

5.2 Robust stability synthesis

Consider the feedback system in Fig.1 with the following plant $G(s, \mathbf{p})$ and PI controller $C(s, \mathbf{x})$:

$$G(s, \mathbf{p}) = \frac{1}{s^2 + p_1 s + p_2}, \quad C(s, \mathbf{x}) = x_1 + \frac{x_2}{s}.$$

Then the characteristic polynomial is

$$g(s, \mathbf{x}, \mathbf{p}) = s^3 + p_1 s^2 + (x_1 + p_2)s + x_2. \tag{20}$$

The nominal value of parameters are $\mathbf{p}^0 = [p_1^0, p_2^0] = [1, 1]$ and the weights are $w_1 = w_2 = 1$. Now we find the feasible set of parameter values of $\mathbf{x}$ for the system to achieve a given level of stability margin $\delta(= \rho^2)$ in Hurwitz sense (*i.e.* $\alpha(t) = \mathbf{i}t$, $\{r_1 = 0\}$). Let $\delta = 0.5$. By Liénard-Chipart criterion $g(s, \mathbf{x}, \mathbf{p}^0)$ is Hurwitz iff

$$\theta(\mathbf{x}) = (x_2 > 0 \ \wedge \ x_1 - x_2 + 1 > 0)$$

holds. The QE problem corresponding to (15) is

$$\forall t > 0 t^8 + (-2x_1 - 2)t^6 + (x_1^2 + 2x_1 + \frac{3}{2})t^4 - 2x_2 t^2 + x_2^2 > 0.$$

By applying QE, we have

$$\phi(\mathbf{x}) = (x_2 < 0 \ \vee \ (P_1 > 0 \wedge x_2 \neq 0)),$$

where

$$P_1(\mathbf{x}) = 256x_2^4 - 768x_1x_2^3 - 768x_2^3 + 16x_1^4x_2^2 + 64x_1^3x_2^2 + 736x_1^2x_2^2 + 1344x_1x_2^2 + 480x_2^2 - 32x_1^5x_2 - 160x_1^4x_2 - 464x_1^3x_2 - 752x_1^2x_2 - 528x_1x_2 - 112x_2 + 8x_1^6 + 48x_1^5 + 124x_1^4 + 176x_1^3 + 142x_1^2 + 60x_1 + 9.$$

For $r_1 = 0$, (8) is inconsistent, hence we have $\rho(0) = \infty$. For non full rank case, (8) is inconsistent. Consequently, the formula

$$\theta(\mathbf{x}) \ \wedge \ \phi(\mathbf{x}) \tag{21}$$

shows the feasible set of parameters $\mathbf{x}$ for the system to achieve a desired level of stability radius. The shaded region in Fig.2 corresponds to (21).

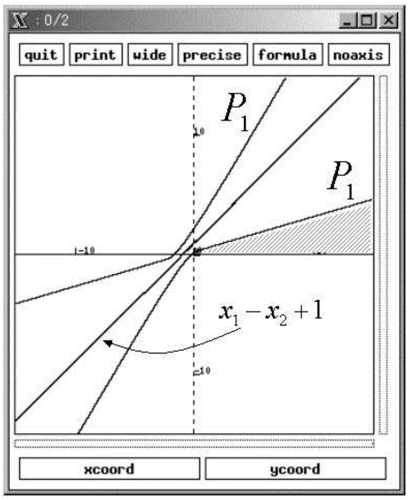

Figure 2: The possible region of $\mathbf{x}$ described by (21)

5.3 Robust stability with sensitivity

We can add any design constraint, which can be reduced to a SDC, in the robust stabilization in §5.2. A typical example is to add finite frequency H_∞ norms of interested closed-loop transfer functions such as sensitivity function $S(s)$. Let us consider the robust stabilization for the same system in §5.2 under a sensitivity constraint

$$||S(s)||_{[0,1]} \equiv \max_{0\leq\omega\leq 1} ||S(\mathbf{i}\omega)|| < 0.1 \tag{22}$$

where

$$S(s) = \frac{s^3+s^2+s}{s^3+s^2+(x_1+1)s+x_2}.$$

We can see from a simple symbolic computation that the frequency restricted H_∞-norm constraint (22) is also reduced to the following SDC:

$$\forall z > 0(x_2^2z^3 + (x_1^2+2x_1+3x_2^2-2x_2-99)z^2 + (2x_1^2+2x_1+3x_2^2-4x_2-99)z + x_1^2+x_2^2-2x_2-99) > 0.$$

Performing QE to this gives the following condition in $\mathbf{x}$:

$$(P_3 > 0 \wedge P_4 \geq 0) \vee (P_2 \geq 0 \wedge P_4 \geq 0), \tag{23}$$

where

$$\begin{aligned}
P_2(\mathbf{x}) &= 3x_2^2 - 2x_2 + x_1^2 + 2x_1 - 99,\\
P_3(\mathbf{x}) &= 264627x_2^4 + 7128x_1x_2^3 - 349668x_2^3 - 3596x_1^3x_2^2 + 169274x_1^2x_2^2 +\\
&462528x_1x_2^2 - 13152942x_2^2 + 2392x_1^4x_2 + 7952x_1^3x_2 - 426492x_1^2x_2 -\\
&705672x_1x_2 + 19405980x_2 - 400x_1^6 - 1996x_1^5 + 105419x_1^4 + 352836x_1^3 -\\
&9467766x_1^2 - 15524784x_1 + 288178803,\\
P_4(\mathbf{x}) &= x_2^2 - 2x_2 + x_1^2 - 99.
\end{aligned}$$

The shaded region in Fig.3 corresponds to (23).

Finally, by superposing (21) and (23) in the parameter space we obtain the admissible region (shaded region in Fig.4) of $\mathbf{x}$ which meets the all requirements given in §5.2, 5.3.

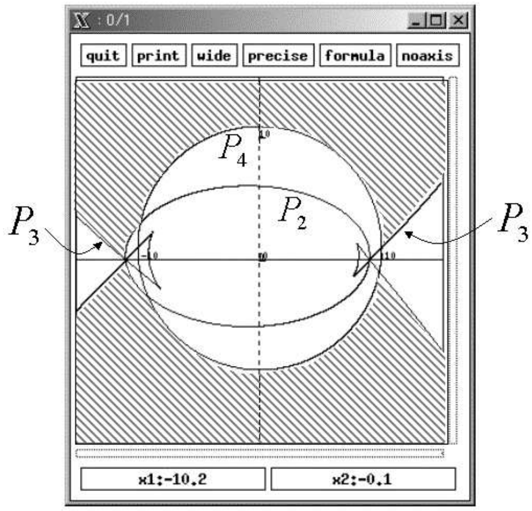

Figure 3: The possible region of $\mathbf{x}$ described by (23)

6 Conclusion

In this paper we proposed a new parameter space design method for robust control synthesis, particularly in terms of stability radius, based on the scheme of combining of SDC and special QE. We showed several concrete examples that demonstrate the validity of our approach. We can accommodate our method naturally to independent perturbation setting among the polynomial coefficients just by using an explicit formula for the case (*e.g.* [5]). In this sense, the framework presented in this paper would provide a unifying platform for further research along this direction.

The advantages of using QE is to be able to resolve many control synthesis problems that are difficult to solve in view of numerical methods. Moreover, QE can be also useful for building up the mathematical modeling (in particular for optimization) of the problems that has no appropriate formularization.

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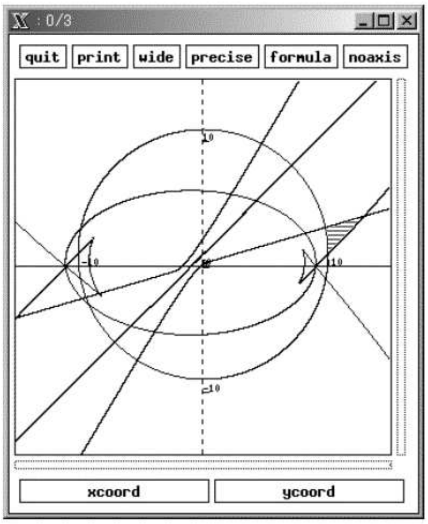

Figure 4: The possible region of $\mathbf{x}$ by (21) $\wedge$ (23)

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