Deformation of Singularity on an Irreducible Quartic Curve by Using the Computer Algebra System Risa/Asir

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1 Introduction

Irreducible quartic curves are classified by the singularities. In this paper, we consider the deformation of irreducible quartic curve by using the computer algebra system Risa/Asir.

Let P^2 be a 2-dimensional complex projective space with the coordinate $[x, y, z]$ and let $f_n(x, y, z)$ be a homogeneous polynomial of degree n in P^2. We consider the set $V_n := \{(x, y, z)|f_n(x, y, z) = 0\}$. We call V_4 a complex projective plane quartic curve but simply a quartic curve throughout this paper. There exist 21 types of curves as the classification of irreducible quartic curves([1]).

Let $f_1, f_2, ..., f_r$ be holomorphic functions defined in an open set U of the complex space C^n. Let X be the analytic set $f_1^{-1}(0) \cap ... \cap f_r^{-1}(0)$. Let $x \in X$, and let $g_1, g_2, ..., g_s$ be a system of generators of ideal $I(X)_{x_0}$ of the generators of the holomorphic functions which vanish identically on a neighborhood of x_0 in X. x_0 is called a simple point of X if the matrix $(\partial g_i/\partial x_j)$ attains its maximal rank. Otherwise, x_0 is called a singular point (singularity) of X. (For $r = 1, x_0$ is called a hypersurface singularity of X.)

Let V be an analytic set in C^n. A singular point x_0 of V is said to be isolated if, for some open neighborhood W of x_0 in C^n, $W \cap V - \{x_0\}$ is a smooth submanifold of $W - \{x_0\}$.

Let (X, x) be a germ of normal isolated singularity of dimension n. Suppose that X is a Stein space. Let $\pi : (M, E) \to (X, x)$ be a resolution of singularity. Then for $1 \le i \le n-1$, $dim(R^i\pi, \vartheta_M)_X$ is finite. $R^i\pi, \vartheta_M$ has support on x. They are independent of the resolution.

We denote them by

$$h^i(X, x) := dim(R^i\pi, \vartheta_M)_X \quad (1 \le i \le n-2)$$

and

$$P_g(X, x) := dim(R^{n-1}\pi, \vartheta_M)_X.$$

The invariant $P_g(X, x)$ is called the geometric genus of (X, x).

Let X be a normal 2-dimensional analytic space. Then the singular points of X are discrete. There are rational singularities, elliptic singularities and so on.

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A singular point x of X is called rational if $P_g(X,x) = 0$. (The singularity (X,x) is also called rational even when dim $X \geq 3$ if the direct image sheaf $R^i\pi, \vartheta_M = 0$ for $i > 0$.) For a rational singularity $x \in X$, the multiplicity of X at x equals $-Z_0^2$ and the local embedding dimension of X at x is $-Z_0^2+1$. Hence a rational singularity with multiplicity 2, which is called a rational double point (A_n, D_n, E_6, E_7, E_8), is a hypersurface singularity ([2]).

On the basis of the above-mentioned theory, we consider the deformation of singularities on a quartic curve. And we make it clear that the structure of singularity changes by a change of parameters of the defining equation.

2 Singularities of quartic curves

For the classification of irreducible quartic curves, the following result is known([1]). (Fundamental type means a class of $P^2 - C$ classified by logarithmic Kodaira dimension.)

Number of singularities	Type of singularities	Number
1	A_6	I_a
1	E_6	I_b
1	A_5	II_a
1	D_5	II_b
1	D_4	$II_{\frac{1}{2}a}$
2	A_4A_2	$II_{\frac{1}{2}b}$
2	A_1A_4	III_a
2	A_3A_2	III_b
2	A_1A_3	III_c
3	$A_2A_2A_2$	III_d
3	$A_2A_2A_1$	III_e
3	$A_2A_1A_1$	III_f
3	$A_1A_1A_1$	III_g
1	A_4	III_h
1	A_3	III_i
2	A_2A_2	III_j
2	A_2A_1	III_k
2	$A_{11}A_1$	III_l
1	A_2	III_m
1	A_1	III_n
0		III_o

3 Deformation of singularity

We consider the following defining equation:

$$f = x^2z^2 \pm 2xy^2z + y^4 + y^3z + a_1yz^3 + a_2z^4 = 0.$$

The curve defined by this equation has a singularity at $[1,0,0]$ in P^2.

$$f_x\mid_{z=1} = 2x + 2y^2,\ f_y\mid_{z=1} = 4xy + 4y^3 + 3y^2 + a_1,\ f_z\mid_{z=1} = 2x^2 + 2xy^2 + y^3 + 3a_1y + 4a_2.$$

Let G be the Grobner Base with lexicographic order for $f_x\mid_{z=1}, f_y\mid_{z=1}, f_z\mid_{z=1}$.

$$G = (-4a_1^3 - 27a_2^2, -9a_2y + 2a_1^2, 2a_1y + 3a_2, 3y^2 + a_1, 3x - a_1)$$

(We calculated the Grobner Base by using the computer algebra system Risa/Asir ([3]))

As a result, the curve defined by $f = 0$ has the only singularity at $[1,0,0]$ for $4a_1^3 + 27a_2^2 \neq 0$. This curve is type III_h.

And the curve defined by $f = 0$ has the A_2 singularity at $[0,0,1]$ for $a_1 = a_2 = 0$. This curve is type $II_{\frac{1}{2}b}$.

We understand that singularity type III_a and singularity type $II_{\frac{1}{2}b}$ occur as a state of deformation of double cusp singularity type III_h

We consider the deformation of irreducible quartic curve with a singularity. In summary, we obtain the following result.

$$f = x^2z^2 \pm 2xy^2z + y^4 + y^3z + a_1yz^3 + a_2z^4 = 0.$$

$$4a_1^3 + 27a_2^2 \neq 0 \rightarrow \text{type } III_h.$$
$$4a_1^3 + 27a_2^2 = 0 \text{ and } \{\ a_1 \neq 0 \text{ or } a_2 \neq 0\} \rightarrow \text{type } III_a.$$
$$a_1 = 0 \text{ and } a_2 = 0 \rightarrow \text{type } II_{\frac{1}{2}b}.$$

Therefore, by the value of parameters of the defining equation, it occurs the new singularity. This is an example of deformation of singularities. It means that the structure of singularity changes by a change of parameters of the defining equation.

参 考 文 献

[1] S.Iitaka, K.Ueno and U.Namikawa, Math. seminar, An extra number, Introduction to modern mathematics [6]: Descartes' spirit and Algebraic Geometry(in Japanese), Nihon Hyoronsha, Tokyo, 1979.

[2] V.I.Arnol'd, Normal forms of functions in neighbourhoods of degenerate critical points, Russian Math. Surveys 29:2, pp. 10-50, 1974.

[3] T.Saito, T.Takeshima and T.Hilano, Risa/Asir Guide Book(in Japanese), SEG shuppan, Tokyo,1998.