

Computing Explicit Formulae for the Radius of Cyclic Hexagons and Heptagons *

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Abstract

This paper describes computations of the circumradius of cyclic polygons given by the lengths of the sides. Extending the results of Robbins (1994) and Pech (2006), for the first time we succeeded in explicitly computing the defining polynomials of the radius of cyclic hexagons and heptagons, with degrees 14 and 38 respectively. We discuss efficient algorithms for elimination by resultants, and characterize the obtained polynomials to confirm their correctness, considering equilateral cases.

1 Introduction

In this study, we consider a classic problem in Euclidean geometry for cyclic polygons; that is, polygons inscribed in a circle. In particular, we focus on computing the circumradius r of cyclic n -gons given by the lengths of sides a_1, a_2, \dots, a_n . Recently, the case for cyclic pentagons was solved by elaborate computations. D. P. Robbins [7] showed that the defining polynomial of r^2 has degree 7, and P. Pech [6] computed the actual form of this polynomial.

In our previous paper [4], it was pointed out that Japanese mathematicians in the 17th century had already derived the identical equation with degree 14 for the circumdiameter of cyclic pentagons, even though the equation itself was not explicitly described. In addition, the author briefly reported the circumradius of cyclic hexagons at the ISSAC2010 poster session [3].

In this paper, we show the details of computations for cyclic hexagons and heptagons to give an efficient algorithm. Using these computations, we have explicitly obtained polynomials that define the circumradius of hexagons and heptagons, with degrees 14 and 38, respectively, as conjectured by Robbins. Moreover, we elucidate the actual forms of these polynomials and confirm their correctness considering degenerated or equilateral cases.

Several authors have studied the “area formula (Heron polynomial)” for cyclic polygons since Robbins [7], as the outline of recent progress on this problem is shown in [5]. Robbins’ conjecture of Equation (13) on the degree of generalized Heron polynomials was proved by M. Fedorchuk

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and I. Pak [1], and later another simpler proof was given by F.M. Maley et al. [2]. Independently of these studies, V.V. Varfolomeev [8] discusses the area and the circumradius of cyclic polygons, but has never obtained an explicit formula for $n > 5$.

In contrast, this paper focuses on the “radius formula” for cyclic polygons, which does not seem to have been so closely investigated in the above papers. To the best of our knowledge, there exist no other reports in which the circumradii of hexagons and heptagons are explicitly computed. However, the result for heptagons is already so huge that it seems impossible to handle cyclic octagons by analogous algorithms using existing computer algebra systems.

2 Previously known results

2.1 Circumradius of cyclic quadrilateral

Firstly, we consider the circumradius r of a triangle with side lengths a_1 , a_2 , and a_3 . It is straightforward to obtain the following relation using cosine and sine rules

$$(a_1^4 + a_2^4 + a_3^4 - 2a_1^2a_2^2 - 2a_2^2a_3^2 - 2a_3^2a_1^2)r^2 + a_1^2a_2^2a_3^2 = 0. \quad (1)$$

In the following, we let $x := r^2$ and consider the defining polynomial in x for each inscribed polygon. From the above equation, we express the defining polynomial for a cyclic triangle as

$$\Phi_3(a_1, a_2, a_3; x) := (a_1^4 + a_2^4 + a_3^4 - 2a_1^2a_2^2 - 2a_2^2a_3^2 - 2a_3^2a_1^2)x + a_1^2a_2^2a_3^2. \quad (2)$$

Solving Equation (1) with r , we obtain the formula of Heron

$$r = \frac{a_1a_2a_3}{\sqrt{(a_1 + a_2 + a_3)(-a_1 + a_2 + a_3)(a_1 - a_2 + a_3)(a_1 + a_2 - a_3)}}. \quad (3)$$

We compute the circumradius of a cyclic $(n+1)$ -gon by recurrently using the result for an n -gon. In this process, we use an auxiliary polynomial F_3 by replacing a_i^2 with b_i in Φ_3 for computational efficiency

$$F_3(b_1, b_2, b_3; x) := (b_1^2 + b_2^2 + b_3^2 - 2b_1b_2 - 2b_2b_3 - 2b_3b_1)x + b_1b_2b_3. \quad (4)$$

Secondly, we divide a given cyclic quadrilateral by a diagonal with length u into two triangles with lengths of sides $\{a_1, a_2, u\}$ and $\{a_3, a_4, u\}$. Since these triangles have a circumcircle in common, we compute the following resultant to eliminate $v(= u^2)$

$$F_4(b_1, b_2, b_3, b_4; x) := \text{Res}_v(F_3(b_1, b_2, v; x), F_3(b_3, b_4, v; x))/x^2, \quad (5)$$

where the redundant factor x^2 is removed. When we let $\Phi_4(a_1, a_2, a_3, a_4; x) := F_4(a_1^2, a_2^2, a_3^2, a_4^2; x)$, this polynomial is factored as follows

$$\begin{aligned} \Phi_4(a_i; x) = & ((-a_1 + a_2 + a_3 + a_4)(a_1 - a_2 + a_3 + a_4)(a_1 + a_2 - a_3 + a_4)(a_1 + a_2 + a_3 - a_4)x \\ & - (a_1a_2 + a_3a_4)(a_1a_3 + a_2a_4)(a_1a_4 + a_2a_3)) \\ & \times ((a_1 + a_2 + a_3 + a_4)(a_1 - a_2 - a_3 + a_4)(a_1 - a_2 + a_3 - a_4)(a_1 + a_2 - a_3 - a_4)x \\ & - (a_1a_2 - a_3a_4)(a_1a_3 - a_2a_4)(a_1a_4 - a_2a_3)). \end{aligned} \quad (6)$$

Solving the first factor, we obtain the classic result of Brahmagupta

$$r = \sqrt{\frac{(a_1a_2 + a_3a_4)(a_1a_3 + a_2a_4)(a_1a_4 + a_2a_3)}{(-a_1 + a_2 + a_3 + a_4)(a_1 - a_2 + a_3 + a_4)(a_1 + a_2 - a_3 + a_4)(a_1 + a_2 + a_3 - a_4)}}. \quad (7)$$

This result means that the polynomial $\Phi_4(a_1, a_2, a_3, a_4; x)$ in Equation (6) defines the circumradii of a cyclic quadrilateral including convex and non-convex cases for a given set of lengths $\{a_1, a_2, a_3, a_4\}$.

2.2 Circumradius of cyclic pentagon

From the computation in the previous subsection, we straightforwardly obtain the following recurrence relation for $n > 3$

$$\begin{cases} F_n(b_1, \dots, b_n; x) := \text{Res}_v(F_{n-1}(b_1, \dots, b_{n-2}, v; x), F_3(b_{n-1}, b_n, v; x))/x^\ell \\ \Phi_n(a_1, \dots, a_n; x) := F_n(a_1^2, \dots, a_n^2; x). \end{cases} \quad (8)$$

We should note that the power ℓ of the redundant factor x^ℓ cannot be predicted before computing the resultant.

Similarly to computing $\Phi_4(a_i; x)$, we compute $\Phi_5(a_i; x)$ for the circumradius of a cyclic pentagon as follows

$$\begin{cases} F_5(b_1, \dots, b_5; x) := \text{Res}_v(F_4(b_1, b_2, b_3, v; x), F_3(b_4, b_5, v; x))/x \\ \Phi_5(a_1, \dots, a_5; x) := F_5(a_1^2, \dots, a_5^2; x) \\ = A_7x^7 + \dots + A_1x + A_0 \quad (A_i \in \mathbf{Z}[a_1, \dots, a_5]). \end{cases} \quad (9)$$

In this paper, we put aside the reduction of $\Phi_n(a_i; x)$ using the symmetry among a_i 's, and focus on the number of terms in the expanded form of the defining polynomials $\Phi_n(a_i; x)$. If we expand all the A_i 's, this $\Phi_5(a_i; x)$ has 2,922 terms in $\mathbf{Z}[a_1, \dots, a_5, x]$.

We characterize the polynomial $\Phi_5(a_i; x)$ by considering special cases of side lengths a_1, \dots, a_5 to confirm its correctness.

- If we consider the case degenerated to a quadrilateral letting $a_5 := 0$, we obtain

$$\Phi_5(a_1, a_2, a_3, a_4, 0; x) = x^3 \{\Phi_4(a_1, a_2, a_3, a_4; x)\}^2. \quad (10)$$

- If we consider the equilateral case, we obtain the following equation

$$\begin{aligned} \Phi_5(1, 1, 1, 1, 1; x) &= 1215x^7 - 3240x^6 + 3618x^5 - 2205x^4 + 795x^3 - 170x^2 + 20x - 1 \\ &= (5x^2 - 5x + 1)(3x - 1)^5 = 0. \end{aligned} \quad (11)$$

Therefore, the radii of the circumcircles are

$$r = \sqrt{\frac{1}{2} + \frac{\sqrt{5}}{10}}, \sqrt{\frac{1}{2} - \frac{\sqrt{5}}{10}}, \frac{1}{\sqrt{3}}, \quad (12)$$

which respectively correspond to the cases of regular pentagon, regular pentagram, and (five degenerated) regular triangles.

3 Circumradius of cyclic hexagon

The degrees of defining polynomials $\Phi_n(a_i; x)$ are conjectured by Robbins [7] and proved by Fedorchuk and Pak [1] as follows. Let

$$k_m := \frac{2m+1}{2} \binom{2m}{m} - 2^{2m-1} = \sum_{j=0}^{m-1} (m-j) \binom{2m+1}{j}; \quad (13)$$

that is, let $k_i := 1, 7, 38, 187, 874, \dots$ ($i = 1, 2, 3, 4, \dots$). Then,

- the degree in x of $\Phi_{2m+1}(a_i; x)$ is k_m , and
- the degree in x of $\Phi_{2m+2}(a_i; x)$ is $2k_m$, where Φ_{2m+2} is factored into the product of two polynomials with each degree k_m .

We computed the case of cyclic hexagon ($m = 2$), using the recurrence relation of Equation (8). As a result, we obtained a polynomial with degree 14 as an explicit form

$$\begin{cases} F_6(b_1, \dots, b_6; x) & := \text{Res}_v(F_5(b_1, b_2, b_3, b_4, v; x), F_3(b_5, b_6, v; x))/x^8 \\ \Phi_6(a_1, \dots, a_6; x) & := F_6(a_1^2, \dots, a_6^2; x) \\ & = B_{14}x^{14} + \dots + B_1x + B_0 = 0 \quad (B_i \in \mathbf{Z}[a_1, \dots, a_6]). \end{cases} \quad (14)$$

This computation required 95 seconds of CPU time in the following environment: Maple14 on Win64, Xeon(2.93 GHz) \times 2, 24 GB RAM.

Next, we factorized $\Phi_6(a_i; x)$, and obtained

$$\Phi_6(a_i; x) = \phi(a_i; x) \cdot \varphi(a_i; x) \quad (\deg_x \phi = \deg_x \varphi = 7), \quad (15)$$

using approximately 9.0 hours of CPU time in the above computational environment. Both ϕ and φ have 19,449 terms and Φ_6 has 497,417 terms, in their expanded forms.

Finally, we characterize the polynomial Φ_6 considering special cases of given lengths a_1, \dots, a_6 as in Section 2.2. From the facts shown below, we believe that the obtained $\Phi_6(a_i; x)$ is the correct polynomial for a cyclic hexagon.

- If we put $a_6 := 0$ in $\phi(a_i; x)$ and $\varphi(a_i; x)$, we have

$$\phi(a_1, \dots, a_5, 0; x) = \varphi(a_1, \dots, a_5, 0; x) = \Phi_5(a_1, \dots, a_5; x); \quad (16)$$

hence, the following relation between Φ_6 and Φ_5 holds

$$\Phi_6(a_1, \dots, a_5, 0; x) = \{\Phi_5(a_1, \dots, a_5; x)\}^2. \quad (17)$$

- If we consider equilateral cases, we have

$$\phi(1, 1, 1, 1, 1, 1; x) = 1024(3x-1)(2x-1)^6, \quad (18)$$

each factor of which respectively corresponds to a regular triangle and a regular square (6-fold), and we also have

$$\varphi(1, 1, 1, 1, 1, 1; x) = 0 \quad (\text{identically}), \quad (19)$$

which means that a regular hexagon cannot be expressed by $\phi(a_i; x)$, $\varphi(a_i; x)$, nor $\Phi_6(a_i; x)$.

The meaning of Equation (19) is interpreted as follows. When n is even, equilateral n -gons contain “pairwise contracted” broken lines with $n/2$ segments. The circumradii of those figures can change continuously ($1/2 \leq r < \infty$), and they are not expressed as roots of any polynomials.

Instead of Equation (19), we can conduct the following analysis. If we put $a_1, \dots, a_5 := 1$ firstly, we obtain the relation

$$\varphi(1, 1, 1, 1, 1, a_6; x)/(a_6 - 1)^{10} = ((a_6 - 5)x^2 + 5x - 1)((a_6 + 3)x - 1)^5. \quad (20)$$

Secondly, if we put $a_6 := 1$ at the right hand side of Equation (20), we have $(4x - 1)^6(x - 1) = 0$. The first factor ($r = 1/2$) corresponds to the case where the six sides are degenerated to only one segment (6-fold), and the second factor ($r = 1$) corresponds to a regular hexagon.

4 Circumradius of cyclic heptagon

In order to compute the circumradius of a cyclic heptagon, we need to compute the following resultant using the recurrence relation of Equation (8)

$$F_7(b_1, \dots, b_7; x) := \text{Res}_v(F_6(b_1, b_2, b_3, b_4, b_5, v; x), F_3(b_6, b_7, v; x))/x^\ell. \quad (21)$$

Unfortunately, this computation seemed too complicated to handle directly. Hence, we took the following steps. First, F_6 and F_3 are expressed as polynomials in v

$$\begin{cases} f(v) := F_6 = p_{16}v^{16} + \dots + p_1v + p_0 & (p_i \in \mathbf{Z}[b_1, \dots, b_5, x]) \\ g(v) := F_3 = q_2v^2 + q_1v + q_0 & (q_i \in \mathbf{Z}[b_6, b_7, x]). \end{cases} \quad (22)$$

Next, instead of $\text{Res}_v(F_6, F_3)$, we compute the pseudo-remainder of $f(v)$ divided by $g(v)$

$$h(v) := \text{prem}_v(f, g) = r_1v + r_0 \quad (r_i \in \mathbf{Z}[b_1, \dots, b_7, x]). \quad (23)$$

Hence, we need to subsequently compute $\text{Res}_v(g, h)$, but these polynomials are still too large to handle with the built-in function for resultants in Maple14. On the other hand, we should note that

$$\text{Res}_z(az^2 + bz + c, dz + e) = \begin{vmatrix} a & b & c \\ d & e & \\ & d & e \end{vmatrix} = ae^2 + cd^2 - bde, \quad (24)$$

which simply expresses the resultant of two polynomials with degrees two and one. Therefore, we obtain

$$\psi(b_i; x) := \text{Res}_v(g, h) = q_2r_0^2 + q_0r_1^2 - q_1r_2r_1, \quad (25)$$

which means that we have eliminated v from $f(v)$ and $g(v)$. This depends on the specification of Maple14 that holds $\psi(b_i; x)$ in an unexpanded form until it is explicitly ordered to expand. These steps required about 434 seconds of CPU time in the same environment shown in Section 3.

Finally, removing the redundant factor, we obtain

$$\begin{cases} F_7(b_1, \dots, b_7; x) := \psi(b_1, \dots, b_7; x)/x^{21} \\ \Phi_7(a_1, \dots, a_7; x) := F_7(a_1^2, \dots, a_7^2; x) \\ = C_{38}x^{38} + \dots + C_1x + C_0 = 0 \end{cases} \quad (C_i \in \mathbf{Z}[a_1, \dots, a_7]). \quad (26)$$

Here we remark that each C_i is so large that we cannot expand $\Phi_7(a_i; x)$ in our computational environment. For example, the maximum number of terms among C_i 's is 19,464,837 for C_{19} . However, we have confirmed that $\Phi_7(a_i; x)$ has degree 38 in x as conjectured by Robbins, and we believe that the obtained $\Phi_7(a_i; x)$ is correct from its characteristics shown below.

- The coefficients have the following structures

$$\begin{cases} C_{38} &= \prod^{64} (a_1 \pm a_2 \pm a_3 \pm a_4 \pm a_5 \pm a_6 \pm a_7) \quad (\text{all combinations}), \\ C_0 &= a_1^{20} a_2^{20} a_3^{20} a_4^{20} a_5^{20} a_6^{20} a_7^{20}. \end{cases} \quad (27)$$

- If we put $a_7 := 0$, we have the following relation between Φ_7 and Φ_6

$$\Phi_7(a_1, \dots, a_6, 0; x) = x^{10} \{\Phi_6(a_1, \dots, a_6; x)\}^2. \quad (28)$$

- If we consider equilateral cases, we have

$$\Phi_7(1, \dots, 1; x) = (7x^3 - 14x^2 + 7x - 1)(5x^2 - 5x + 1)^7(3x - 1)^{21}. \quad (29)$$

The first factor $7x^3 - 14x^2 + 7x - 1$ is derived from the formula for a septimal angle

$$\sin 7\theta = -64 \sin^7 \theta + 112 \sin^5 \theta - 56 \sin^3 \theta + 7 \sin \theta, \quad (30)$$

for $\theta = \pi/7$ and $\sin \theta = 1/(2r)$. Therefore, this factor represents a regular heptagon and two types of star-like heptagons. The second factor corresponds to a regular pentagon and pentagram (7-fold each), and the third factor corresponds to a regular triangle (21-fold).

5 Concluding remarks

n	Degree	No. of terms
3	1	7
4	2	71
5	7	2,922
6	14	497,417
7	38	337,550,051

Table 1: Defining polynomial $\Phi_n(a_i; x)$ of circumradius of cyclic n -gon

In this study, we succeeded in computing explicit formulae for the circumradius of cyclic hexagons and heptagons for the first time, and investigated the characteristics of each defining polynomial. We summarize the shapes of $\Phi_n(a_i; x)$ for $n = 3, \dots, 7$ in Table 1. The degrees 14 for a hexagon and 38 for a heptagon coincide rightly with those conjectured by Robbins. As a result, we believe that it is a significant breakthrough to have obtained $\Phi_6(a_i; x)$ and $\Phi_7(a_i; x)$ in explicit forms from the viewpoint of practical computation.

If we try to compute the circumradius of a cyclic octagon, we have to compute the following resultant

$$F_8(b_1, \dots, b_8; x) := \text{Res}_v(F_7(b_1, b_2, b_3, b_4, b_5, b_6, v; x), F_3(b_7, b_8, v; x))/x^\ell. \quad (31)$$

However, F_7 is already so huge (nearly 15 GB) that it seems impossible to handle octagons by a similar approach to that presented in Section 4 using existing computer algebra systems.

References

- [1] Fedorchuk, M. and Pak, I.: Rigidity and Polynomial Invariants of Convex Polytopes, *Duke Math. J.*, **129**(2), 2005, 371–404.
- [2] Maley, F. M., Robbins, D. P., and Roskies, J.: On the Areas of Cyclic and Semicyclic Polygons, *Advances in Applied Mathematics*, **34**(4), 2005, 669–689.
- [3] Moritsugu, S.: Radius Computation for an Inscribed Pentagon in *Sanpou-Hakki* (1690), *ACM Communications in Computer Algebra*, **44**(3), 2010, 127–128.
- [4] Moritsugu, S.: Radius Computation for Inscribed Polygons and Its Solution in *Sanpou-Hakki* (1690), *Kyoto University RIMS Kokyuroku*, 2011. (in press; in Japanese).
- [5] Pak, I.: The Area of Cyclic Polygons: Recent Progress on Robbins' Conjecture, *Advances in Applied Mathematics*, **34**(4), 2005, 690–696.
- [6] Pech, P.: Computations of the Area and Radius of Cyclic Polygons Given by the Lengths of Sides, *ADG2004* (Hong, H. and Wang, D., eds.), *LNAI*, **3763**, Gainesville, Springer, 2006, 44–58.
- [7] Robbins, D. P.: Areas of Polygons Inscribed in a Circle, *Discrete & Computational Geometry*, **12**(1), 1994, 223–236.
- [8] Varfolomeev, V. V.: Inscribed Polygons and Heron Polynomials, *Sbornik: Mathematics*, **194**(3), 2003, 311–331.