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論文

The exceptional set for the projection from the moduli space of polynomials

Masayo FUJIMURA*
Department of Mathematics, National Defense Academy

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Abstract

The natural projection from the moduli space of polynomials of degree n is not surjective if $n \geq 4$. We give explicit parametric representation of the exceptional set when $n = 4$ and 5. And we describe degeneration which occurs above the exceptional set when $n = 4$. Also we show that the preimage of a point generally consists of $(n-2)!$ points, where $(n-2)!$ is the maximum when the preimage is a finite set.

1 Known results

Let Poly_n be the space of all polynomial maps of degree n:

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0, \quad (a_j \in \mathbb{C}\ (j = 1, \cdots, n),\ a_n \neq 0).$$

Let $\mathfrak{A}$ be the group of all affine transformations. We say that two maps $p_1,\ p_2 \in \mathrm{Poly}_n$ are *affine conjugate*, denoted by $p_1 \sim_{\mathfrak{A}} p_2$, if there exist a $g \in \mathfrak{A}$ with $g \circ p_1 \circ g^{-1} = p_2$. The *moduli space* of polynomial maps degree n is the set of all affine conjugacy classes of elements in Poly_n, which is denoted by M_n.

For each $f \in \mathrm{Poly}_n$, let $z_1, z_2, \cdots, z_{n+1}$ be the fixed points of f and μ_j the multipliers at z_j; $\mu_j = f'(z_j)$ $(1 \leq j \leq n+1)$, we set $z_{n+1} = \infty$ and hence $\mu_{n+1} = 0$. The elementary symmetric functions of μ_j are

$$\begin{gathered}\sigma_{n,1} = \mu_1 + \mu_2 + \cdots + \mu_{n+1},\ \cdots,\ \sigma_{n,r} = \sum_{j_1 < j_2 < \cdots < j_r} \mu_{j_1}\mu_{j_2}\cdots\mu_{j_r},\ \cdots, \\ \sigma_{n,n+1} = \mu_1\mu_2\cdots\mu_{n+1} (= 0).\end{gathered} \tag{1}$$

Note that these quantities are invariant under affine conjugacy.

*masayo@nda.ac.jp

Next, the *holomorphic index* of a rational function f at a fixed point $\zeta \in \mathbb{C}$ is defined to be the complex number

$$\iota(f,\zeta) = \frac{1}{2\pi i}\int_C \frac{dz}{z - f(z)}$$

where we integrate in a small loop C in the positive direction around ζ (see [5]). The following facts are well known as Fatou's index theorem:

- If the multiplier $\mu \neq 1$, then $\iota(f,\zeta) = \frac{1}{1-\mu}$.
- For any polynomial map f which is not the identity map,

$$\sum_{j=1}^{n} \iota(f, z_j) = 0. \tag{2}$$

In particular, we obtain the following linear relation among the elementary symmetric functions $\sigma_{n,j}$ $(1 \leq j \leq n+1)$:

$$0 = \sigma_{n,n+1} = \sum_{k=0}^{n-1} (-1)^{n-k-1}(n-k)\sigma_{n,k} \tag{3}$$

where we put $\sigma_{n,0} = 1$. (See Theorem 1 in [4]).

Hence, we have a natural projection Ψ_{Poly_n} from a point in M_n to an $(n-1)$-tuple $(\sigma_{n,1}, \sigma_{n,2}, \cdots, \sigma_{n,n-2}, \sigma_{n,n}) \in \mathbb{C}^{n-1}$:

$$\Psi_{\mathrm{Poly}_n} : \mathrm{M}_n \longrightarrow \mathbb{C}^{n-1}.$$

And in [2] we showed that Ψ_{Poly_n} is not surjective if $n \geq 4$.

Theorem 1 ([2])
The exceptional set

$$\mathcal{E}(n) = \mathbb{C} \setminus \Psi_{\mathrm{Poly}_n}(\mathrm{M}_n)$$

is nonempty for every $n \geq 4$.

To state the situation more precisely, we define the following subset.

Definition 2
Let $\Sigma_*(n)$ $(\subset \mathbb{C}^{n-1})$ be the set of points $(s_{n,1}, \cdots, s_{n,n-2}, s_{n,n})$ such that the corresponding solutions $\{m_1, m_2, \cdots, m_n\}$ of (1) with $\sigma_{n,j} = s_{n,j}$ for every j, where $\sigma_{n,n-1}$ is defined by (3), satisfies one of the following conditions A, B, and C, where we set $\Omega = \{1, \cdots, n\}$.

Condition A

1. $m_j \neq 1$ $(\forall j \in \Omega)$,
2. $\sum_{j\in\Omega} \frac{1}{1-m_j} = 0$, and
3. for any proper subset ω of Ω, $\sum_{j\in\omega} \frac{1}{1-m_j} \neq 0$.

Condition B Let Ω' be the set $\{k \in \Omega\,;\, m_k \neq 1\}$ and N the cardinality of Ω'.

1. $1 \leq N \leq n-2$, and
2. for any subset ω' of Ω', $\sum_{j\in\omega'} \frac{1}{1-m_j} \neq 0$.

Condition C $m_j = 1$ $(\forall j \in \Omega)$.

Remark 3
The set $\Sigma_*(n)$ is disjoint union of the set of points satisfying the conditions A, B, and C, which in turn denoted by $X_A,\ X_B$, and X_C, respectively.

Then $\Sigma_*(n)$ is contained in the image of Ψ_{Poly_n}, i.e., $\Sigma_*(n) \cap \mathcal{E}(n) = \emptyset$.

Theorem 4 ([2])
For any point $(s_{n,1},\ s_{n,2},\ \cdots,\ s_{n,n-2},\ s_{n,n})$ in $\Sigma_*(n)$, there exists a polynomial of degree n having n values $s_{n,1},\ s_{n,2},\ \cdots,\ s_{n,n-2},\ s_{n,n-1},\ s_{n,n}$ as the elementary symmetric functions of the multipliers at the fixed points.

Also, we showed the following theorem.

Theorem 5 ([2])
There does not exist a polynomial of degree N with the following multipliers at the fixed points;

$$\underbrace{m,\cdots,m,\frac{n_1-m}{n_1-1}}_{n_1},\ \underbrace{m,\cdots,m,\frac{n_2-m}{n_2-1}}_{n_2},\ \cdots,\underbrace{m,\cdots,m,\frac{n_k-m}{n_k-1}}_{n_k}\quad (m \neq 1,\ k \geq 2), \tag{4}$$

where $n_j \geq 2$ $(j = 1, 2, \cdots, k)$ and $N = \sum_{j=1}^{k} n_j$.

2 On the exceptional set

If $n = 4$, we can parameterize $\mathcal{E}(4)$ as follows.

Theorem 6 ([4])
The exceptional set $\mathcal{E}(4)$ is a punctured curve in $\mathbb{C}^3$, and the defining equation is given by:

$$(\sigma_1, \sigma_2, \sigma_4) = \left(4,\ s,\ \frac{(s-4)^2}{4}\right), \quad s \in \mathbb{C} \setminus \{6\}. \tag{5}$$

That is, none of quartic polynomials corresponds to the multipliers $\mu, \mu, 2-\mu, 2-\mu$ $(\mu \neq 1)$.

Moreover the following theorem clarify degeneration of polynomials from dynamical viewpoint when points in $\mathbb{C}^3$ tend to the exceptional set $\mathcal{E}(4)$ with real s in (5).

Theorem 7
Let D be a subset of $\Sigma(4) = \Psi_{\mathrm{Poly}_n}(\mathrm{M}_n)$ defined by

$$D = \left\{(4, s_2, s_4) \mid s_2 < -\frac{1}{4}(s_4^2 - 6s_4 - 19),\ s_4 < \frac{(2-s_2)^2}{4}\right\} \subset \{4\} \times \mathbb{R}^2.$$

For any $\sigma \in \mathbf{D}$, let p_σ be an element in Poly_3 corresponding to σ. Then we can construct two polynomial-like maps (see [1]) $(U, V, p_\sigma) \equiv z^2 + c$ and $(\tilde{U}, \tilde{V}, p_\sigma) \equiv z^2 + \bar{c}$ so that c and $\bar{c}$ converge to a common value $\tilde{c} \in \mathbb{R}$ as σ tends to a point of $\mathcal{E}(4)$.

The limit value $\tilde{c}$ depends only on the landing point $(4, s, \frac{(s-4)^2}{4}) \in \mathcal{E}(4)$ and is written as $\tilde{c} = \frac{s-4}{8}$.

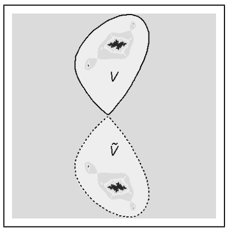

Figure 1:

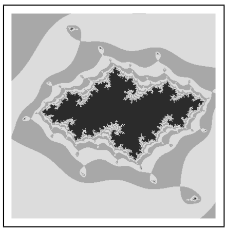

Figure 2:

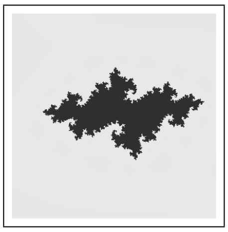

Figure 3:

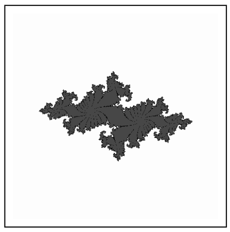

Figure 4:

Figure 1 shows Julia set of $p(z) = z^4 + 3.8199z^2 + z + 3.775218$ that corresponds to the point $(4, -1.7696160, 8.8480801) \in \Sigma(4)$. $(-2 < \Re z,\ \Im z < 2.)$

Figure 2 shows enlargement of Figure 1. $(-0.2 < \Re z < 0.28,\ 1.137 < \Im z < 1.617.)$

Figure 3 shows Julia set of corresponding quadratic-like map. $(-0.2 < \Re z < 0.28,\ 1.137 < \Im z < 1.617.)$

Figure 4 shows Julia set of quadratic polynomial $p_c(z) = z^2 + (-0.726 + 0.183i)$.

Proof From (5), $\mathcal{E}(4)$ is contained in the plane $\{(4, \sigma_2, \sigma_4)\} \cong \mathbb{C}^2$. On D, any corresponding polynomial p_σ has two attracting fixed points of multiplier μ, $\overline{\mu}$ and the three critical points $x_0 \in \mathbb{R}$, $z_0, \bar{z}_0 \in \mathbb{C} \setminus \mathbb{R}$. Dynamics of p_σ are symmetric with respect to the real axis (see Figure 1). Hence we can choose suitable topological disks V, $\widetilde{V}$ bounded by equipotential curves such that $z_0 \in V, \bar{z}_0 \in \widetilde{V}$ and $V \cap \widetilde{V} = \emptyset$. Then $(f(V), V, p_\sigma)$ and $(f(\widetilde{V}), \widetilde{V}, p_\sigma)$ are quadratic-like maps hybrid equivalent to $z^2 + c$ and $z^2 + \bar{c}$ respectively. If σ converges to a point $(4, s, \frac{(s-4)^2}{4}) \in \mathcal{E}(4)$, two parameters $c, \bar{c}$ are converge to common value $\frac{s-4}{8}$ (see Figure 3 and 4). ∎

Next, even when $n = 5$, we can give the explicit parametric representation of the exceptional set $\mathcal{E}(5)$.

Theorem 8
The exceptional set $\mathcal{E}(5)$ is parameterized as follows:

$$
\begin{aligned}
&(\sigma_1, \sigma_2, \sigma_3, \sigma_5) \\
&\quad = \left(s, \tfrac{-4s^2+76s-190}{9}, \tfrac{-2(s-8)(4s-5)(2s-7)}{27}, \tfrac{(2s-13)(s-8)(2s-7)^3}{243}\right), \quad (s \in \mathbb{C} \setminus \{5\}).
\end{aligned}
$$

Namely none of polynomials of degree five corresponds to the multipliers $\mu,\ \mu,\ \mu,\ 2-\mu,\ \frac{3-\mu}{2}$ $(\mu \neq 1)$.

Proof For a monic and centered polynomial $p(z) = z^5 + c_3z^3 + c_2z^2 + c_1z + c_0$, the four values $\sigma_{5,1}, \sigma_{5,2}, \sigma_{5,3}, \sigma_{5,5}$ are determined from $c_0, \cdots, c_3$, which can be written down explicitly as follows:

$$
\begin{aligned}
\sigma_{5,1} &= 4c_3^2 - 15c_1 + 20, \\
\sigma_{5,2} &= 4c_3^4 - (36c_1 - 52)c_3^2 + 27c_2^2c_3 - 50c_0c_2 + 80c_1^2 - 220c_1 + 150, \\
\sigma_{5,3} &= (-12c_1 + 24)c_3^4 + 4c_2^2c_3^3 + (40c_0c_2 + 88c_1^2 - 284c_1 + 220)c_3^2 - ((117c_1 - 198)c_2^2 \\
&\quad + 125c_0^2)c_3 + 27c_2^4 + (300c_0c_1 - 450c_0)c_2 - 160c_1^3 + 720c_1^2 - 1050c_1 + 500, \\
\sigma_{5,5} &= 108c_0^2c_3^5 + ((-72c_0c_1 + 72c_0)c_2 + 16c_1^3 - 48c_1^2 + 36c_1)c_3^4 + (16c_0c_2^3 + (-4c_1^2
\end{aligned}
$$

$$+8c_1)c_2^2 - 900c_0^2c_1 + 900c_0^2)c_3^3 + (825c_0^2c_2^2 + (560c_0c_1^2 - 1120c_0c_1 + 600c_0)c_2$$
$$-128c_1^4 + 512c_1^3 - 680c_1^2 + 300c_1)c_3^2 - ((630c_0c_1 - 630c_0)c_2^3 - (144c_1^3 - 432c_1^2$$
$$+315c_1)c_2^2 + 3750c_0^3c_2 - 2000c_0^2c_1^2 + 4000c_0^2c_1 - 1875c_0^2)c_3 + 108c_0c_2^5 - (27c_1^2$$
$$-54c_1)c_2^4 + (2250c_0^2c_1 - 2250c_0^2)c_2^2 - (1600c_0c_1^3 - 4800c_0c_1^2 + 4500c_0c_1$$
$$-1250c_0)c_2 + 256c_1^5 - 1280c_1^4 + 2400c_1^3 - 2000c_1^2 + 625c_1 + 3125c_0^4.$$

Hence, the defining equation of $\mathcal{E}(5)$ can be written as in the theorem. ∎

3 Outside the exceptional set

Outside the exceptional set $\mathcal{E}(n)$, the preimage of a point can contain infinite number of points in general. But if the preimage contains only a finite number of points, we can easily see that it contains at most $(n-2)!$ points. Furthermore, we show the following theorem.

Theorem 9
For every σ in general position, $\Psi_{\mathrm{Poly}_n}^{-1}(\sigma)$ consists of $(n-2)!$ points.

Proof By recalling the definition of the set X_A, the assertion follows from Bézout's theorem. ∎

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