Some Dynamical Loci of Quartic Polynomials

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Abstract

Let $M_4(\mathbb{C})$ be all affine conjugacy classes of quartic polynomials. We define a projection $\Psi_4$ from $M_4(\mathbb{C})$ to $\mathbb{C}^3$ via the elementary symmetric functions of the multipliers of the fixed points. In \cite{2}, we show $\Psi_4$ is not surjective. The image of $M_4(\mathbb{C})$ under $\Psi_4$ is denoted by $\Sigma(4)$. The complement $E(4) = \mathbb{C}^3 \setminus \Sigma(4)$ is called the exceptional set. On a part of the real section of $E(4)$, we verify that a quartic polynomial degenerates into “twins” of quadratic polynomials. We conjecture that this phenomena holds on $E(4)$.

1 Introduction

Let $\text{Poly}_4(\mathbb{C})$ be the space of all quartic polynomials, and $M_4(\mathbb{C})$ be the space of all affine conjugacy classes of quartic polynomials. We define a projection $\Psi_4$ from $M_4(\mathbb{C})$ to $\mathbb{C}^3$ via the elementary symmetric functions of the multipliers of the fixed points. In \cite{2}, we show the projection is not surjective. The image of $M_4(\mathbb{C})$ under $\Psi_4$ is denoted by $\Sigma(4)$, the complement $\mathbb{C}^3 \setminus \Sigma(4)$ by $E(4)$. For the cubic (resp. quadratic) polynomials, the exceptional set is empty. Unfortunately it can happen that the exceptional set $E(n)$ is nonempty for $n \geq 4$ (see \cite{2} and \cite{3}).

This paper consists of two parts, one is devoted to defining an algebraic variety $G(c)$, the other to analyzing dynamics of $\text{Poly}_4(\mathbb{C})$ on neighborhood of $E(4)$.

First, we can define an algebraic variety $G(c)$, given in Section 2, that indicates essential property of the projection $\Psi_4$. We can derive defining equations of the exceptional set and of the branch locus from the perspective of $G(c)$. We mainly use the symbolic and algebraic computation system Risa/Asir to obtain an algebraic variety $G(c)$.

Second, we examine dynamical behavior on the parameter space $\Sigma(4) \cup E(4)$. We have Theorem \ref{thm:16} in Section 5.

\textbf{Theorem \ref{thm:16}}

There is a component $D \subset \Sigma(4)$ such that two polynomial-like maps $(U, V, p) \sim_{hb} z^2 + c$ and $(\tilde{U}, \tilde{V}, \tilde{p}) \sim_{hb} \tilde{z}^2 + \tilde{c}$ are constructed for any $(p) \in D$, and $c$ and $\tilde{c}$ converge to a common value $\tilde{c} \in \mathbb{R}$ as $(p) \to E(4)$. The limit value $\tilde{c}$ depends only on the landing point $(4, s, \frac{i(s-4)}{4}) \in E(4)$ and is written by $\tilde{c} = \frac{s^4}{8}$.

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Because the parameter values (corresponds to the “multipliers”) on $E(4)$ forms $\mu, \mu, 2 - \mu, 2 - \mu$, the following conjectures Conjecture 15 and Conjecture 17 seem very likely.

**Conjecture [15]**
On the exceptional set, a quartic polynomial degenerates into “twins” of quadratic polynomials conjugate to $z^2 + c$ for some $c$.

**Conjecture [17]**
None of quartic polynomial $p$ has two disjoint quadratic-like restrictions of $p$ such that both quadratic-like map are hybrid equivalent to a common quadratic polynomial $z^2 + c$, $c \in M \setminus \{1/2\}$, where $M$ is Mandelbrot set.

These conjectures, given in Section 5, back with the reason why the exceptional set is not empty.

## 2 Definitions and Notations

Let $\text{Poly}_4(\mathbb{C})$ be the space of all polynomials of the form:

$$p(z) = a_4z^4 + a_3z^3 + a_2z^3 + a_1z + a_0 \quad (a_4 \neq 0).$$

Two maps $p_1, p_2 \in \text{Poly}_4(\mathbb{C})$ are *holomorphically conjugate*, denoted by $p_1 \sim p_2$, if and only if there exists $g \in \mathbb{A}(\mathbb{C})$ with $g \circ p_1 \circ g^{-1} = p_2$, where $\mathbb{A}(\mathbb{C})$ is the group of all affine transformations.

The space, $\text{Poly}_4(\mathbb{C})/$, of holomorphic conjugacy classes $(p)$ of quartic polynomials is denoted by $M_4(\mathbb{C})$.

For each $p(z) \in \text{Poly}_4(\mathbb{C})$, let $z_1, \cdots, z_4, z_5 = \infty$ be the fixed points of $p$, and $\mu_1, \cdots, \mu_4, \mu_5 = 0$ the multipliers of $z_i$ (i.e. $\mu_i = p'(z_i)$). Let $\sigma_1, \sigma_2, \cdots, \sigma_5$ be the elementary symmetric functions of these multipliers

$$
\begin{align*}
\sigma_1 &= \mu_1 + \mu_2 + \mu_3 + \mu_4, \\
\sigma_2 &= \mu_1\mu_2 + \mu_1\mu_3 + \mu_1\mu_4 + \mu_2\mu_3 + \mu_2\mu_4 + \mu_3\mu_4, \\
\sigma_3 &= \mu_1\mu_2\mu_3 + \mu_1\mu_2\mu_4 + \mu_1\mu_3\mu_4 + \mu_2\mu_3\mu_4, \\
\sigma_4 &= \mu_1\mu_2\mu_3\mu_4, \\
\sigma_5 &= 0.
\end{align*}
$$

These multipliers are *invariant* under the action of (conjugation) $\mathbb{A}(\mathbb{C})$.

The holomorphic index of a rational function $f$ at a fixed point $\zeta \in \mathbb{C}$ is defined to be the complex number

$$
\iota(f, \zeta) = \frac{1}{2\pi i} \oint_{\zeta} \frac{dz}{z - f(z)},
$$

where we integrate in a small loop in the positive direction around $\zeta$.

The following results are well known as “Fatou’s index theorem”:

- If $\zeta$ is a fixed point of multiplier $\mu \neq 1$, then $\iota(f, \zeta) = \frac{1}{1 - \mu}$.
- For any polynomial $p$ which is not the identity map,

$$
\sum_{\zeta \in \mathbb{C}} \iota(p, \zeta) = 0, \quad (1)
$$

where this summation is over all fixed points of $p$. 
By an automorphism of a polynomial map $p$ we will mean an affine transformation $g$ that commutes with $p$. The collection $\text{Aut}(p)$ of all automorphisms of $p$ forms a finite group. The symmetry locus of the polynomials of degree $n$ is defined to be the set $S_n (\subset M_n)$ consisting of all conjugacy classes $(p)$ of polynomial maps admitting non-trivial automorphisms.

A polynomial-like map of degree $d$ is a triple $(U, V, f)$ where $U$ and $V$ are topological disks, with $V$ relatively compact in $U$, and $f : V \to U$ is analytic, proper of degree $d$.

An orientation-preserving homeomorphism $f$ of a domain $D$ onto another, $D'$, is called a quasiconformal map if $f$ is ACL on every rectangle

$$R = \{z = x + iy | a \leq x \leq b, \quad c \leq y \leq d\},$$

i.e.,

- $f(x + iy)$ is absolutely continuous on $[a, b]$ with respect to $x$ for almost every fixed $y$,
- $f(x + iy)$ is absolutely continuous on $[c, d]$ with respect to $y$ for almost every fixed $x$, and
- there is a constant $k < 1$ such that $|f(x)| \leq k|f(y)|$ almost everywhere on $D$.

The filled-in Julia set $K_f$ of a polynomial-like map $(U, V, f)$ is defined by

$$K_f = \bigcap_{n \geq 0} f^{-n}(V).$$

Polynomial-like maps $(U, V, f)$ and $(\tilde{U}, \tilde{V}, \tilde{f})$ are hybrid equivalent $f \sim_{hb} \tilde{f}$, if there exists a quasi-conformal map $h$ from a neighborhood of $K_f$ to a neighborhood of $K_{\tilde{f}}$, such that $h \circ f = \tilde{f} \circ h$ near $K_f$ and $\partial h = 0$ almost everywhere on $K_f$.

From Straightening Theorem in [1], every polynomial-like map $(U, V, f)$ of degree $d$ is hybrid equivalent to a polynomial $P$ of degree $d$. If $K_f$ is connected then $P$ is unique up to conjugation by an affine map.

## 3 The projection from $M_4(\mathbb{C})$ to $\mathbb{C}^3$

Let $\sigma_1, \cdots, \sigma_4$ be the elementary symmetric functions of the multipliers of $p(z) \in \text{Poly}_4(\mathbb{C})$. The following relation holds by Fatou’s index theorem.

**Lemma 1 (Theorem 1 in [2])**

Among $\sigma_i$’s, there is a linear relation

$$4 - 3\sigma_1 + 2\sigma_2 - \sigma_3 = 0.$$  

For a monic and centered quartic polynomial $z^4 + c_2z^2 + c_1z + c_0$, the three values $\sigma_1, \sigma_2, \sigma_4$ are determined by Transformation formula:

\[
\begin{align*}
\sigma_1 &= -8c_1 + 12, \\
\sigma_2 &= 4c_2^2 - 16c_0c_2 + 18c_1^2 - 60c_1 + 48, \\
\sigma_4 &= 16c_0c_4 + (-4c_1^2 + 8c_1)c_2 - 128c_0^2c_2^2 + (144c_0c_4^2 - 288c_0c_1 + 128c_0)c_2 \\
&\quad - 27c_1^2 + 108c_1^3 - 144c_1^3 + 64c_1 + 256c_0^3.
\end{align*}
\]
Risa/Asir commands for getting Transformation formula:

\[ P = z^4 + c_2 z^2 + c_1 z + c_0; \]
\[ \text{Fix1} = z^0 + z^1 + z^2 + z^3; \]
\[ \text{Fix2} = z^0 (z^1 + z^2 + z^3) + z^1 (z^2 + z^3) + z^2 z^3 - c_2; \]
\[ \text{Fix3} = z^0 z^1 z^2 + z^0 z^1 z^3 + z^0 z^2 z^3 + z^1 z^2 z^3 + c_1 - 1; \]
\[ \text{Fix4} = z^0 z^1 z^2 z^3 - c_0; \]
\[ \text{DP} = \text{diff}(P, z); \]
\[ \text{M0} = \text{subst} (\text{DP}, z, z^0) - m_0; \]
\[ \text{M1} = \text{subst} (\text{DP}, z, z^1) - m_1; \]
\[ \text{M2} = \text{subst} (\text{DP}, z, z^2) - m_2; \]
\[ \text{M3} = \text{subst} (\text{DP}, z, z^3) - m_3; \]
\[ S_1 = m_0 + m_1 + m_2 + m_3 - s_1; \]
\[ S_2 = m_0 (m_1 + m_2 + m_3) + m_1 (m_2 + m_3) + m_2 m_3 - s_2; \]
\[ S_4 = m_0 m_1 m_2 m_3 - s_4; \]
\[ \text{load("gr");} \]
\[ G = \text{gr}([\text{Fix1}, \text{Fix2}, \text{Fix3}, \text{Fix4}, \text{M0}, \text{M1}, \text{M2}, \text{M3}, \text{S1}, \text{S2}, \text{S4}], \]
\[ [m_0, m_1, m_2, m_3, z^0, z^1, z^2, z^3, s_4, s_2, s_1, c_2, c_1, c_0], \]
\[ [[[0, 4], [0, 4], [0, 3], [0, 3]]]; \]

There is a natural projection

\[ \Psi_4 : \mathbb{M}_4(\mathbb{C}) \longrightarrow \Sigma(4) \]

where \( \Sigma(4) \) is the image of \( \mathbb{M}_4(\mathbb{C}) \) under \( \Psi_4 \). The complement \( \mathbb{C}^3 \setminus \Sigma(4) \) is denoted by \( \mathcal{E}(4) \), and called the exceptional set.

Under the conjugacy of the action of \( \mathfrak{A}(\mathbb{C}) \), it can be assumed that any quartic polynomial is “monic” and “centered”:

\[ p(z) = z^4 + c_2 z^2 + c_1 z + c_0. \]

There are three monic and centered polynomials in any conjugacy class \( \langle p \rangle \) except for \( \langle z^4 + z \rangle \), and they are transformed each other under the action of \( G_3 = \{ 1, \omega, \omega^2 \} \), where \( \omega \) is third roots of unity.

After following procedures 1 – 4, we obtain a parametrized algebraic variety. This variety indicates essential property of the projection \( \Psi_4 \).

1. for a point \( \langle p \rangle \in \mathbb{M}_4(\mathbb{C}) \), choose a monic and centered representative \( z^4 + c_2 z^2 + c_1 z + c_0 \),

2. getting rid of the affine ambiguity on “Transformation formula”, set \( c := c_2 \) (if \( c_2 = 0 \), set \( \tilde{c} := c_0 \)), and

3. rebuild Transformation formula of \( \sigma_1, \sigma_2, \sigma_4, c, c_0, c_1 \) variables,

4. remove two variables \( c_0, c_1 \), from the above formula.
**Definition 2**
We define an algebraic variety in $\mathbb{C}^3$ with a parameter $c \in \mathbb{C}$,

$$G(c) : 262144(\sigma_1 - 4)^2c^2 + 1024(27\sigma_1^4 + (144\sigma_2 - 576)\sigma_1^2 + (384\sigma_2 + 1280)\sigma_1 + 128\sigma_2^2$$
$$-256\sigma_2 - 512\sigma_4 - 768)c^2 + (9\sigma_1^2 + 24\sigma_4 - 32\sigma_2 - 48)^3 = 0.$$ 

**Risa/Asir commands for getting $G(c)$:**

```plaintext
Sgm1 = -8*c1 + 12 - s1;
Sgm2 = 4*c2^3 - 16*c0*c2 + 18*c1^2 - 60*c1 + 48 - s2;
Sgm4 = 16*c0*c2^4 + (-4*c1^2 + 8*c1)*c2^3 - 128*c0^2*c2^2 + (144*c0*c1^2 - 288*c0*c1 + 128*c0)*c2 - 27*c1^4 + 108*c1^3 - 144*c1^2 + 64*c1
+ 256*c0^3 - s4;
CC = c2^3 - c;
load("gr");
G1 = gr([Sgm1, Sgm2, Sgm4, CC], [c0, c1, c2, c, s1, s2, s4], [[0, 3], [0, 1], [0, 3]]);
```

The number of conjugacy classes corresponding to a point $(\sigma_1, \sigma_2, \sigma_4) \in \mathbb{C}^3$ is equal to the number of allowable parameter values $c$ on $G(c)$. Namely, we have the following theorem by counting the number of solution $c$ of the defining equation of $G(c)$.

**Theorem 3**
For $(\sigma_1, \sigma_2, \sigma_4) \in \mathbb{C}^3$, number of the elements of set $\Psi_4^{-1}(\sigma_1, \sigma_2, \sigma_4)$ are $\infty$, $0$, $1$ or $2$:

**Case 1** $\#\Psi_4^{-1}(\sigma_1, \sigma_2, \sigma_4) = \infty$ if and only if $(\sigma_1, \sigma_2, \sigma_4) = (4, 6, 1)$. And further, we can precisely formulate,

$$\Psi_4^{-1}(4, 6, 1) = \left\{ p_a(z) = (z^2 - a)^2 + z \right\}_{a \in \mathbb{C}} \quad \text{(note } p_a \sim p_{s(a)}).$$

**Case 2** $\#\Psi_4^{-1}(\sigma_1, \sigma_2, \sigma_4) = 0$ if and only if the point $(\sigma_1, \sigma_2, \sigma_4)$ cannot belong to $G(c)$ for any $c$. And further, we can give defining equation of the exceptional set $E(4)$,

$$\langle \sigma_1, \sigma_2, \sigma_4 \rangle = \left( 4, s, \frac{(s - 4)^2}{4} \right), \quad s \neq 6. \quad (2)$$

**Case 3** $\#\Psi_4^{-1}(\sigma_1, \sigma_2, \sigma_4) = 1$ if and only if discriminant of the defining equation of $G(c)$ vanishes or $\sigma_1 = 4$.

**Case 4** $\#\Psi_4^{-1}(\sigma_1, \sigma_2, \sigma_4) = 2$, for the remains of the above.

Theorem leads immediately to the following two corollaries which are justified for structural and topological reasons.

**Corollary 4**
The exceptional set $E(4)$ is contained in the plane $\{(4, \sigma_2, \sigma_4) \} \cong \mathbb{C}^2$. 
Corollary 5
None of quartic polynomial has the fixed points of the multipliers \( \mu, \mu, 2 - \mu, 2 - \mu, (\mu \neq 1) \).

Corollary 5 is clear from (2) and the relation between roots and coefficients.

Risa/Asir commands for getting the multipliers:
\[
S1=m0+m1+m2+m3-4 ;
S2=m0*(m1+m2+m3)+m1*(m2+m3)+m2*m3-s;
S4=m0*m1*m2*m3-(s-4)^2/4;
Index=nm(1/(1-m0)+1/(1-m1)+1/(1-m2)+1/(1-m3));
load("gr");
G2=gr([S1,S2,S4,Index],[s,m3,m2,m1,m0],2);
\]

Remark 6
The discriminant of the defining equation of \( G(c) \) is given as follows:
\[
\text{Discr} = 1073741824(54\sigma_1^5 + (-81\sigma_2 - 27\sigma_4 - 135)\sigma_1^4 + (36\sigma_2^2 - 144\sigma_2 - 1008)\sigma_1^3 \\
+ (-4\sigma_1^2 + 360\sigma_2^2 + (144\sigma_4 + 2976)\sigma_2 + 576\sigma_4 + 4192)\sigma_1^2 \\
+ (-160\sigma_2^2 - 2176\sigma_2^2 + (-384\sigma_2 - 6400)\sigma_2 - 1280\sigma_2 + 5376)\sigma_1 + 16\sigma_2^4 + 448\sigma_2^3 \\
+ (-128\sigma_4 + 2176)\sigma_2^2 + (256\sigma_4 + 3840)\sigma_2 + 256\sigma_2 + 768\sigma_4 + 2304).
\]
Therefore \( \text{Discr} = 0 \) is a surface in \( \mathbb{C}^3 \).

Theorem 7
The symmetry locus can be formulated as follows:
\[
\begin{align*}
\sigma_1 &= s, \\
\sigma_2 &= 3(3s - 4)(s + 4)/32, \quad (s \in \mathbb{C}). \\
\sigma_4 &= -(3s - 4)^3(s - 12)/4096.
\end{align*}
\] (3)

Proof
The symmetry locus is the singular part of the surface defined by \( \text{Discr} = 0 \) (see [2]).
\[
\text{Discr} = \text{Discr}_{s_1} = \text{Discr}_{s_2} = \text{Discr}_{s_4} = 0, \quad \sigma_1 \neq 4.
\]

Solving these equations we have (3).

Remark 8
The point \((4, 6, 1)\) is the unique intersection point of the symmetry locus and the plane \( \sigma_1 = 4 \).

4 Loci Per\(_1\)(\(\mu\)) on the space \(\{(4, s_2, s_4)\}\)

In this section, we consider dynamical behavior on the real section \(\mathbb{R}^2 \cong \{(4, s_2, s_4)\}\).

The locus Per\(_1\)\((\mu)\) be the set of all conjugacy classes \(\langle p \rangle\) of maps \(p\) having a fixed point of multiplier \(\mu\).
Proposition 9
For each \( \mu \in \mathbb{C} \), \( \text{Per}_1(\mu) \) is a straight line with the following defining equation:

\[
\text{Per}_1(\mu) : \sigma_4 - (2\mu - \mu^2)\sigma_2 + \mu^4 - 4\mu^3 + 8\mu = 0.
\]

Proof  The multipliers at the fixed points are the roots of the equation,

\[
\mu^4 - \sigma_1\mu^3 + \sigma_2\mu^2 - \sigma_3\mu + \sigma_4 = 0.
\]

From the linear relation of Lemma 1, we have the defining equation of \( \text{Per}_1(\mu) \).

We remark that the multipliers of a quartic polynomial on the real plane \( \{(4, \sigma_2, \sigma_4)\} \) are 'four real values', 'two real and a pair of complex conjugates', or 'two pair of complex conjugates'.

4.1 \( \text{Per}_1(\mu) \) (\( \mu \in \mathbb{R} \))

At first we consider \( \mu \in \mathbb{R} \). In this case we can illustrate the figure of \( \text{Per}_1(\mu) \). (See Figure 1) The following results are easily verified.

Proposition 10
For \( (p) \in \{(4, \sigma_2, \sigma_4)\} \cap \Sigma(4) \), the corresponding multipliers of \( p \) are \( \mu, 2 - \mu, \lambda, 2 - \lambda \).

Proof  It is clear from the relation between roots and coefficients.

Risa/Asir commands for getting the multipliers:

\[
\begin{align*}
S1 &= m0 + m1 + m2 + m3 - 4; \\
S2 &= m0^8(m1 + m2 + m3) + m1^8(m2 + m3) + m2^8m3 - s2; \\
S4 &= m0^8m1^8m2^8m3 - s4; \\
\text{Index} &= \text{mm}(1/(1-m0)+1/(1-m1)+1/(1-m2)+1/(1-m3)); \\
\text{load} &= \text{"gr"}; \\
G3 &= \text{gr}([S1, S2, S4, Index], [s2, s4, m3, m2, m1, m0], 2);
\end{align*}
\]

The left figure shows \( \text{Per}_1(\mu) \) \((-10 < \mu < 1)\):
- \(-20 < s_2, s_4 < 20\),
- Gray lines mean \( \text{Per}_1(\mu) \) \((|\mu| \geq 1)\) and black lines mean \( \text{Per}_1(\mu) \) \((|\mu| < 1)\).

Corollary 11
- If \( p \) has a attracting fixed point then \( p \) has a repelling fixed point with positive multiplier.
- If \( p \) has a repelling fixed point with negative multiplier then \( p \) has a repelling fixed point with positive multiplier.

Namely, each line of Figure 1 is overlapped by a line \( \text{Per}_1(\mu) \) for some \( \mu > 1 \), and \( p \) cannot have three attracting fixed points.
4.2 Per₁(µ) and Per₁(˜µ)

Next, we consider the multipliers of a quartic polynomial are 'two real and a pair of complex conjugates'. In this case, the multipliers are \(1 \pm iβ, λ\), and \(2 - λ\) from Proposition[10]. Then we have the following from Proposition[9].

Proposition 12

For each \(β \in \mathbb{R}\), Per₁(\(1 \pm iβ\)) is a straight line with the following defining equation:

\[
\text{Per₁}(1 \pm iβ) : \quad σ₄ = (1 + β²)σ₂ - (1 + β²)(5 + β²).
\]

Proof Removing \(λ\) from two equations \(σ₂ = 5 + β² + λ(2 - λ)\) and \(σ₄ = (1 + β²)λ(2 - λ)\), we have the above defining equation of Per₁(\(1 \pm iβ\)).

Note that these loci are corresponds to repelling fixed points.

Now, we consider the last case: multipliers of a quartic polynomial are 'two pair of complex conjugates'. In this case, the multipliers are \(a \pm ib\) and \(2 - a \pm ib\) from Proposition[10]. Because defining equation of Per₁(µ) can express a line on the real plane no longer, we need a new device Per₁(˜µ) for illustrating figures of Per₁(µ). (See Figure 2)

The locus Per₁(˜µ) be the set of all conjugacy classes (\(p\)) of maps \(p\) having a fixed point of multiplier \(µ\) with \(˜µ = \bar{µ}\).

The left figure shows Per₁(\(1 \pm iβ\)) and Per₁(˜µ).

-20 < s₂, s₄ < 20,

Dark gray lines mean Per₁(\(1 \pm iβ\)),

gray curves mean Per₁(˜µ), \(t ≥ 1\) and

black curves mean Per₁(˜µ), \(t < 1\).

Proposition 13

In the case that the multipliers are \(a \pm ib\) and \(2 - a \pm ib\), we have a defining equation of Per₁(˜µ).

\[
\text{Per₁}(˜µ) : \quad σ₂ = 2(µ² + 2µ)σ₄ + µ³ - 4µ³ + (σ₂ - 16)µ² = 0,
\]

where \(µ = a^2 + b²\).

Figure 2:

Proof In this case the multipliers are \(a \pm ib\) and \(2 - a \pm ib\). By setting \(µ = a^2 + b²\) for two equations \(σ₂ = -2a² + 4a + 2b²\) and \(σ₄ = (a² + b²)(2 - a)² + b²\), we have

\[
σ₂ = -4a² + 4a + 2µ, \quad σ₄ = µ(µ - 4a + 4).
\]

where \(σ₂ < 0\) and \(σ₄ < (\sigma₄ - 6σ₆ - 19)\).

Removing \(a\) from the above two equations, we have a defining equation of Per₁(˜µ).

Remark 14

If \(0 ≤ t < 1\), Per₁(µ) corresponds to polynomials having two attracting fixed points of multiplier \(a + ib\) and \(a - ib\). As \(a, b \in \mathbb{R}\), the discriminant \(4 + 4(4 + 2µ - σ₂)\) of the left of (4) must be positive. Therefore, on a region \{(\(4, σ₂, σ₄\)|\(σ₂ < -\frac{1}{4}(σ₄ - 6σ₆ - 19), σ₄ < \frac{(σ₄ - 6σ₆ - 19)}{4\rangle\)}, corresponding polynomial \(p\) have two attracting fixed points of multipliers \(a \pm ib\).
5 Dynamics on the exceptional set

The lines \( \{ \text{Per}_1(\mu) \} \) have a close relation with the exceptional set. As an example, we give the following results directly obtained by the results in the section \[4.1\] and \[4.2\].

The left figure shows the real section of the exceptional set

\[
E(4) : \left( 4, s, \frac{(s - 4)^2}{4} \right), \quad (s \neq 6).
\]

- On the plane \( \{(4, s_2, s_4)\} \cong \mathbb{R}^2 \), the envelopes of the lines \( \{ \text{Per}_1(\mu) \}_{\mu \in \mathbb{R}} \) and of \( \{ \text{Per}_1(1 + i\beta) \}_{\beta \in \mathbb{R}} \) coincides with the exceptional set. (See Figure 1 and 2.)

- On the region \( \{ (4, \sigma_2, \sigma_4) \mid \sigma_4 < \frac{2 - \sigma_2^2}{4} \} \) that bounded by the exceptional set, corresponding quartic polynomial has the fixed points of the multiplier with two pair of complex conjugates.

**Conjecture 15**

On the exceptional set, a quartic polynomial degenerates into “twins” of quadratic polynomials conjugate to \( z^2 + c \) for some \( c \).

**Theorem 16**

There is a component \( D \subset \Sigma(4) \) such that two polynomial-like maps \( (U, V, p) \sim_{hb} z^2 + c \) and \( (\tilde{U}, \tilde{V}, \tilde{p}) \sim_{hb} \tilde{z}^2 + \tilde{c} \) are constructed for any \( (p) \in D \), and \( c \) and \( \tilde{c} \) converge to a common value \( \tilde{c} \in \mathbb{R} \) as \( (p) \to E(4) \). The limit value \( \tilde{c} \) depends only on the landing point \( (4, s, \frac{(s - 4)^2}{4}) \in E(4) \) and is written by \( \tilde{c} = c - \frac{4}{s} \).

Proof On a region \( \{ (4, \sigma_2, \sigma_4) \mid \sigma_2 < -\frac{1}{4}(\sigma_4^2 - 6\sigma_4 - 19), \sigma_4 < \frac{2 - \sigma_2^2}{4} \} \), any corresponding polynomial \( p(z) \) has two attracting fixed points of multiplier \( \mu, \tilde{\mu} \). Dynamics of \( p(z) \) are symmetry for the real axis. (See Figure 3) Therefore we can choose suitable topological disk \( U, \tilde{U} \) bounded by equipotential curves such that \( (U, V, p) \) and \( (\tilde{U}, \tilde{V}, \tilde{p}) \) \((U \cap \tilde{U} = \emptyset) \) are quadratic-like maps hybrid equivalent to \( z^2 + c \) and \( \tilde{z}^2 + \tilde{c} \) respectively.

Then, if \( (p) \) converges to a point \( (4, s, \frac{(s - 4)^2}{4}) \in E(4) \), two parameters \( c \) and \( \tilde{c} \) are converges to common value \( c - \frac{4}{s} \in \mathbb{R} \). (See Figure 6 and 7 Figure 8, 9 show another example.)

**Figure 3:**

**Figure 4:** (4, -1.7696160, 8.8480801), Julia set of \( p(z) = z^4 + 3.8199z^2 + z + 3.775218, -2 < \Re z, \Im z < 2 \)

**Figure 5:** Julia set of \( p(z) = z^4 + 3.8199z^2 + z + 3.775218, -0.2 < \Re z < 0.28, 1.137 < \Im z < 1.617 \)
Figure 6: Julia set of quadratic-like map $-0.2 < \Re z < 0.28, 1.137 < \Im z < 1.617$

Figure 7: Julia set of $p_c(z) = z^2 + (-0.726 + 0.183i), -2 < \Re z, \Im z < 2.$

Figure 8: (4, 3, 7), Julia set of $p(z) = z^4 + 0.62996z^2 + z + 0.39685, -2 < \Re z, \Im z < 2.$

Figure 9: Julia set of $p(z) = z^4 + 0.62996z^2 + z + 0.39685, -0.5 < \Re z < 0.8, 0 < \Im z < 1.3.$

Figure 10: Julia set of quadratic-like map $-0.5 < \Re z < 0.8, 0 < \Im z < 1.3$

Figure 11: Julia set of $p_c(z) = z^2 + (-0.125 + 0.65i), -2 < \Re z, \Im z < 2.$
6 Dynamics on the point \((4, 6, 1) \in \Sigma(4)\)

One parameter family \(\{p_a(z) = (z^2 - a)^2 + a\}_{a \in \mathbb{C}}\) \((p_a \sim p_{x_0})\) corresponds to the point \((4, 6, 1)\). There are maps \(p\) in this family such that \(p\) have two disjoint quadratic-like restriction, hybrid equivalent to common quadratic map \(z^2 + \frac{1}{2}\). (See Figure 12 and 13)

Figure 12: Julia set of \(p(z) = z^4 - z^2 + z + 1\), \(-2 < \Re z, \Im z < 2, (4, 6, 1) \in \Sigma(4)\)

Figure 13: Julia set of \(p(z) = z^4 + 2z^2 + z + 1\), \(-2 < \Re z, \Im z < 2, (4, 6, 1) \in \Sigma(4)\)

Figure 14: Julia set of \(p(z) = z^4 - z^2 + z + 0.25\), \(-2 < \Re z, \Im z < 2, (4, 6, 1) \in \Sigma(4)\)

Figure 15: Julia set of \(p(z) = z^4 + z\), \(-2 < \Re z, \Im z < 2, (4, 6, 1) \in \Sigma(4)\)

On the other hand, in Figure 14 the largest Fatou components contains two critical points. Therefore in this case \(p\) cannot have two disjoint quadratic-like restriction. The quartic polynomial in Figure 15 has unique parabolic fixed point at the origin.

Conjecture 17

None of quartic polynomial \(p\) have two disjoint quadratic-like restrictions of \(p\) such that both quadratic-like map are hybrid equivalent to a common quadratic polynomial \(z^2 + c, c \in \mathbb{M} \setminus \{\frac{1}{4}\}\), where \(\mathbb{M}\) is Mandelbrot set.

This conjecture gives a reason why the exceptional set is not empty.
References


