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# Defining equations and rigidity of 3-dimensional quotient terminal singularities—a computational approach—

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**Abstract**

We compute the defining equations of 3-dimensional quotient terminal singularities in the case where the order of the acting cyclic group is small (less than or equal to 9), utilizing the algebraic computation system MAGMA. Then we show the rigidity of such singularities by directly calculating $T^1$ by means of the defining equations thus obtained. The calculation of $T^1$ is done by the computer algebra system SINGULAR.

## 1 Introduction

According to Minimal Model Program for 3-dimensional projective varieties established by Mori, Kawamata and others, minimal models do exist in the category of 3-dimensional normal projective varieties which are $\mathbf{Q}$-factorial and admit at worst *terminal* singularities (see [5] for instance). Terminal singularities in dimension 3 are classified by Mori ([7]). Roughly speaking, they are classified into two types: one is the quotient of 3-dimensional affine space $\mathbf{C}^3$ by a finite cyclic group (*quotient terminal singularity*) and the other is the quotient of a hypersurface in $\mathbf{C}^4$ by a finite cyclic group.

The purpose of this note is

(i) to compute the defining equations of 3-dimensional quotient terminal singularities in the case where the order of the cyclic group is small (less than or equal to 9) (**Main Result 1**)

(ii) to show the rigidity of such singularities by directly calculating $T^1$ utilizing the defining equations obtained in (i) (**Main Result 2**).

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We note that the rigidity of isolated 3-dimensional (or higher dimensional) quotient singularities in general was established by Schlessinger [9] and well-known for specialists. Our own interest lies in the calculation of explicit defining equations and $T^1$ of 3-dimensional quotient terminal singularities, through which we can appreciate the power of such excellent algebraic computation systems as MAGMA and SINGULAR. We use MAGMA for the calculation of the fundamental invariants and their relations with respect to a given cyclic group action, and also use SINGULAR for the calculation of $T^1$.

We finally note that most of the contents of this letter are taken from the master thesis of the second author [11].

## 2 Review of Quotient Terminal Singularities and Rigidity of Isolated Quotient Singularities

Throughout this note, we are concerned with the 3-dimensional quotient terminal singularities. For a general reference on algebraic geometry, see [4]. For the theory of deformation of singularities, [1] is a standard text. By Terminal Lemma ([8, p.34]), it is known that any 3-dimensional quotient terminal singularity is explicitly described as follows:

let $r, a$ be coprime positive integers with $r \geq 2$, $1 \leq a < r$. Let $\zeta_r := e^{2\pi i/r}$ be a $r$-th root of unity and set

$$g := \begin{pmatrix} \zeta_r^a & 0 & 0 \\ 0 & \zeta_r^{-a} & 0 \\ 0 & 0 & \zeta_r \end{pmatrix} \in \mathbf{GL}(3, \mathbf{C}).$$

We define a cyclic group

$$G(r, a) := \langle g \rangle = \{I_3, g, g^2, \cdots, g^{r-1}\} \subset \mathbf{GL}(3, \mathbf{C}).$$

Then any 3-dimensional quotient terminal singularity is isomorphic locally to the isolated singularity at the origin of the quotient $X(r, a) := \mathbf{A}^3/G(r, a) = \text{Spec } \mathbf{C}[x, y, z]^{G(r,a)}$, where $G(r, a)$ acts linearly on $\mathbf{A}^3$ and $\mathbf{C}[x, y, z]^{G(r,a)}$ is the invariant ring of the polynomial ring $\mathbf{C}[x, y, z]$ with respect to this $G(r, a)$-action. We temporarily call $X(r, a)$ as the (3-dimensional) quotient terminal singularity of type $(r, a)$ in this note.

Now we review some well-known results on deformations of affine varieties. Let $X_0$ be a given affine variety and $X, S$ schemes with $P \in S$ a closed point. We call a flat morphism $\varphi : X \to S$ with closed fiber $\varphi^{-1}(P) \cong X_0$ a deformation of $X_0$. A deformation of $X_0$ with base space $S = \text{Spec } \left(\mathbf{C}[x]/\langle x^2 \rangle\right)$, where $\mathbf{C}[x]/\langle x^2 \rangle$ is the ring of dual numbers, is called a first order deformation of $X_0$. The set of all isomorphism classes of first order deformations of $X_0$ can be computed as follows:

let $X_0 \to \mathbf{A}^n$ be a given closed embedding and $\Theta_{X_0}$ the tangent sheaf (module) of $X_0$. We have a natural exact sequence

$$0 \to \Theta_{X_0} \to \Theta_{\mathbf{A}^n}|_{X_0} \to N_{X_0},$$

where $\Theta_{\mathbf{A}^n}|_{X_0}$ is the restriction of $\Theta_{\mathbf{A}^n}$ to $X_0$ and $N_{X_0}$ is the normal sheaf (module) of $X_0$ in $\mathbf{A}^n$. We define

$$T^1_{X_0} := \mathrm{coker}(\Theta_{\mathbf{A}^n}|_{X_0} \to N_{X_0}).$$

Then it is well-known that the set of all isomorphisum classes of first order deformations of $X_0$ is canonically isomorphic to $T^1_{X_0}$ ([1, Theorem 6.2]).

Now, we say by definition that $X_0$ is *rigid* if $X_0$ has no nontrivial first order deformations. Equivalently, $X_0$ is rigid iff $T^1_{X_0} = \{0\}$. The next theorem due to Schlessinger shows the rigidity of isolated quotient singularities .

**Theorem 1 ([9])**
*Let $X$ be a smooth affine variety of dimension greater than or equal to three and $G$ a finite automorphism group of $X$. Suppose that $G$ has a unique fixed point $x \in X$ and no nontrivial element of $G$ leaves fixed any point in $X$ other than $x$. Let $p : X \to Y := X/G$ be the quotient of $X$ by $G$ so that $Y$ has a unique singularity at $y := p(x)$. Then $Y$ is rigid.*

From the above theorem, it follows immediately that a 3-dimensional quotient terminal singularity is rigid. In Section 3 below, we will directly calculate and show $T^1_{X(r,a)} = \{0\}$ for small $r$ $(1 \leq r \leq 9)$.

# 3 Main results

Let $G(r,a) \subset \mathbf{GL}(3,\mathbf{C})$ be as in Section 2 and let $R(r,a) := \mathbf{C}[x,y,z]^{G(r,a)}$ $\subset \mathbf{C}[x,y,z]$ be the invariant ring of $G(r,a)$. Then $R(r,a)$ is the affine ring of the 3-dimensional quotient singularity $X(r,a)$ of type $(r,a)$. Let $\{f_1, f_2, \cdots, f_m\} \subset R(r,a)$ be a minimal generating set for $R(r,a)$ as a $\mathbf{C}$-algebra. Then we may write $R(r,a) = \mathbf{C}[f_1, f_2, \cdots, f_m]$. We define a surjective homomorphism $\varphi : \mathbf{C}[X_1, X_2, \cdots, X_m] \to R(r,a)$ by $\varphi(X_i) := f_i$ $(1 \leq i \leq m)$. Let $I(r,a) := \mathrm{Ker}\ \varphi \subset \mathbf{C}[X_1, \cdots, X_m]$ be the relation ideal for $\{f_1, f_2, \cdots, f_m\}$. Then we call a generating set of $I(r,a)$ as a set of defining equations of $X(r,a)$.

**Main Result 1**
*We computed the invariant ring $R(r,a)$ and a generating set of $I(r,a)$ explicitly when $r \leq 9$.*

For lack of space, we omit giving the table of all $R(r,a)$ and $I(r,a)$ for $r \leq 9$. Instead, we concentrate on the case $(r,a) = (5,2)$ as a typical example.

Suppose $(r,a) = (5,2)$. Then $R(5,2)$ and $I(5,2)$ is given as follows:

$$R(5,2) = \mathbf{C}[xy, x^2z + yz^2, x^2z, xz^3, y^3z, x^5 + y^5 + z^5, x^5, y^5] \subset \mathbf{C}[x,y,z]$$

$$I(5,2) = \langle a^5 - gh, a^3c - eg, a^3d - c^2e, a^2cd + 2adg - b^2g + c^2g, a^3f - a^3g - a^3h - cde, a^2b - a^2c - ce, a^2d - 1/2ab^2 + 1/2bg - 1/2cg + 1/2de, a^2f - a^2g - a^2h + acd - 1/3b^3 + 1/3dg + 1/3ef - 1/3eg - 1/3eh, ac^2 - bg + cg, c^3 - dg, c^2d - fg + g^2 + gh, a^2e - ch, abe - ace - dh, ade - 1/2b^2e + 1/2c^2e + 1/2fh - 1/2gh - 1/2h^2, ad - bc + c^2, af - ag - ah - bd + cd, cf - cg - ch - d^2, bh - ch - e^2\rangle \subset \mathbf{C}[a,b,c,d,e,f,g,h],$$

where we denote the generators $\{xy, x^2z + yz^2, \cdots\}$ of $R(5,2)$ by $\{a, b, \cdots\}$ from the left. We reproduce the MAGMA session of the computation of $R(5,2)$ and $I(5,2)$ for readers' convenience. It is as follows:

```
> F<z> := CyclotomicField(5);
> G52 := MatrixGroup<3,F|[z^2,0,0,0,z^(-2), 0,0,0,z]>;
> R52 := InvariantRing(G52);
> print PrimaryInvariants(R52);
[ x1*x2,  x1^2*x3 + x2*x3^2,  x1^5 + x2^5 + x3^5 ]
> print SecondaryInvariants(R52);
[ 1,  x1^2*x3, x1*x3^3, x2^3*x3, x1^5, x2^5 ]
> print FundamentalInvariants(R52);
[ x1*x2, x1^2*x3 + x2*x3^2, x1^2*x3, x1*x3^3, x2^3*x3,
  x1^5 + x2^5 + x3^5, x1^5, x2^5 ]
> Q := RationalField();
> P<x1,x2,x3,x4,x5> := PolynomialRing(Q,5);
> S := [x1*x2,x1^2*x3 + x2*x3^2, x1^2*x3, x1*x3^3,x2^3*x3,
x1^5 + x2^5 + x3^5, x1^5,x2^5];
> P8<a,b,c,d,e,f,g,h> := PolynomialRing(Q,[2,3,3,4,4,5,5,5]);
> P8;
Graded Polynomial ring of rank 8 over Rational Field
Lexicographical Order
Variables: a, b, c, d, e, f, g, h
Variable weights: 2 3 3 4 4 5 5 5
> I52:= RelationIdeal(S,P8);
> print I52;
Ideal of Graded Polynomial ring of rank 8 over Rational Field
```

```
Lexicographical Order
Variables: a, b, c, d, e, f, g, h
Variable weights: 2 3 3 4 4 5 5 5
Basis:
[ a^5 - g*h,
  a^3*c - e*g, a^3*d - c^2*e, a^2*c*d + 2*a*d*g - b^2*g + c^2*g,
  a^3*f - a^3*g - a^3*h - c*d*e,  a^2*b - a^2*c - c*e,
  a^2*d - 1/2*a*b^2 + 1/2*b*g - 1/2*c*g + 1/2*d*e,
  a^2*f - a^2*g - a^2*h + a*c*d - 1/3*b^3 + 1/3*d*g + 1/3*e*f
          - 1/3*e*g - 1/3*e*h,  a*c^2 - b*g + c*g, c^3 - d*g,
  c^2*d - f*g + g^2 + g*h, a^2*e - c*h, a*b*e - a*c*e - d*h,
  a*d*e - 1/2*b^2*e + 1/2*c^2*e + 1/2*f*h - 1/2*g*h - 1/2*h^2,
  a*d - b*c + c^2, a*f - a*g - a*h - b*d + c*d,
  c*f - c*g - c*h - d^2, b*h - c*h - e^2 ]
```

**Remark 1**

*We give a few comments on the above session.* " > " *is the* MAGMA *prompt. In the first line, we set* $F := \mathbf{Q}(z)$*, where* $\mathbf{Q}$ *is the field of rational numbers and* $z = e^{\frac{2\pi i}{5}}$*. The second line registers* $G52 = G(5,2) = \langle \begin{pmatrix} z^2 & 0 & 0 \\ 0 & z^{-2} & 0 \\ 0 & 0 & z \end{pmatrix} \rangle \subset \mathbf{GL}(3,F)$*. The third line registers the invariant ring* $R52 = R(5,2)$*. Primary, secondary and fundamental invariants of* $R(5,2)$ *mean the following. Generally, let* $R$ *be the invariant ring of a finite group* $G \subset \mathbf{GL}(n,\mathbf{C})$*. Then it is known that* $R$ *is a graded Cohen-Macaulay ring and it admits a Hironaka decomposition*

$$R = \oplus_{i=1}^{t} \mathbf{C}[\alpha_1, \cdots, \alpha_n] \cdot \beta_i,$$

*where* $\{\alpha_1, \cdots, \alpha_n\}$ *is homogeneous and algebraically independent over* $\mathbf{C}$ *and* $R$ *is a finitely generated free* $\mathbf{C}[\alpha_1, \cdots, \alpha_n]$*-module with basis* $\{\beta_1, \cdots, \beta_t\}$ *(cf. [10]). We call* $\{\alpha_1, \cdots, \alpha_n\}$ *primary invariants and* $\{\beta_1, \cdots, \beta_t\}$ *secondary invariants of* $R$*. Fundamental invariants of* $R$ *are a minimal set of generators of* $R$ *chosen from the union of the primary and secondary invariants of* $R$*. For more details on* MAGMA*, see [2].*

**Main Result 2**

*For* $r \leq 9$*,* $T^1_{X(r,a)} = \{0\}$*. In particular,* $X(r,a)$ *is rigid.*

For the computations of $T^1_{X(r,a)}$, we use SINGULAR. Here we reproduce the SINGULAR session for $(r,a) = (5,2)$ as an example.

```
> LIB "sing.lib";
// ** loaded /Singular/LIB/1-2-2/sing.lib (1.13,1998/06/02)
```

```
// ** loaded /Singular/LIB/1-2-2/random.lib (1.5,1998/05/05)
// ** loaded /Singular/LIB/1-2-2/general.lib (1.8.2.1,1998/09/30)
// ** loaded /Singular/LIB/1-2-2/inout.lib (1.6,1998/05/14)
> ring R = 0,(a,b,c,d,e,f,g,h), ds;
> R;
//   characteristic : 0
//   number of vars : 8
//        block   1 : ordering ds
//                  : names    a b c d e f g h
//        block   2 : ordering C
> ideal I = a^5-g*h, a^3*c-e*g,a^3*d-c^2*e,
a^2*c*d+2*a*d*g-b^2*g+c^2*g, a^3*f-a^3*g-a^3*h-c*d*e,
a^2*b-a^2*c-c*e,
a^2*d-(1/2)*a*b^2+(1/2)*b*g-(1/2)*c*g+(1/2)*d*e,
a^2*f-a^2*g-a^2*h+a*c*d-(1/3)*b^3+(1/3)*d*g+(1/3)*e*f
-(1/3)*e*g-(1/3)*e*h, a*c^2-b*g+c*g, c^3-d*g,
c^2*d-f*g+g^2+g*h, a^2*e-c*h, a*b*e-a*c*e-d*h,
a*d*e-(1/2)*b^2*e+(1/2)*c^2*e+(1/2)*f*h-(1/2)*g*h-(1/2)*h^2,
a*d-b*c+c^2,a*f-a*g-a*h-b*d+c*d, c*f-c*g-c*h-d^2,b*h-c*h-e^2;
> kbase(T1(I));
// dim T1 = 0
_[1]=0
> quit;
```

**Remark 2**
*In the above session, the first command [> LIB "sing.lib";] calls the* Singular *library "sing.lib" which deals with singularity computation. The second command [> ring R = 0,(a,b,c,d,e,f,g,h), ds;] declares the polynomial ring* $R := \mathbf{Q}[a, b, c, d, e, f, g, h]$ *with local degree reverse lexicographical ordering. The fourth command [> ideal I = $\cdots$;] registers our ideal* $I(5, 2)$*. The fifth command [> kbase(T1(I));] gives the base of* $T^1_{I(5,2)}(= T^1_{X(5,2)})$ *as* $\mathbf{Q}$*-vector space.* Singular *calculates* $T^1_X$ *as* $\mathrm{coker}(\Theta_{\mathbf{A}^n}|_X \to N_X)$*. For the details on the calculation of* $T^1$ *by* Singular*, see [6, Section 3].*

# 4 Concluding Remarks

In this note, we computed the defining equations of the 3-dimensional quotient terminal singularity $X(r, a)$ of type $(r, a)$ for small $r$ $(r \le 9)$ and showed their rigidity by explicitly calculating $T^1_{X(r,a)} = \{0\}$, using Magma and Singular. We could not compute the $r \ge 10$

case with our system (Intel Pentium III (1GHz), Memory 512MB, Windows 2000). To analyze how much it costs to compute the equations of $X(r,a)$ and $T^1_{X(r,a)}$ for larger $r$ will be an interesting problem. But we here leave it as a future problem to be discussed.

As stated in Introduction, a 3-dimensional terminal singularity other than quotient ones is a quotient of a hypersurface in $\mathbf{C}^4$ by a finite cyclic group. In this case, the moduli space (namely the base space of the miniversal deformation) is expected to be nontrivial. SINGULAR has a program written by B. Martin [6], by which it is possible to calculate the base and total space of miniversal deformation space of such isolated singularities. Hence it will be interesting to know the explicit equations of the moduli spaces of these terminal singularities. If we do, then we will be able to answer such questions as:
(i) given a terminal singularity, to which type of terminal singularities can it deform?
(ii) which type of singularity does the base space of the miniversal deformation have?
These are of quite interest from the viewpoint of algebraic geometry.

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