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論文

Symbolic Algorithms for Obtaining Moments and “Moments of Moments” of Bootstrap Distributions

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Abstract

Symbolic procedures for expressing the moments of bootstrap distributions in terms of multivariate version of symmetric polynomials in the observations and their moments with respect to population (*i.e.*, moments of bootstrap moments) in terms of population moments, respectively, are proposed with theoretical discussion. Two examples of application of the procedures to the bootstrap distributions of the sample correlation coefficient and of the sample regression coefficient, demonstrate their practical utility.

1 Introduction

The *bootstrap* method proposed by Efron [7] is a versatile computer-intensive statistical method for estimating the sampling moments of a statistic based on the given sample drawn from a population with less information on its properties than required in traditional statistical analyses. This method has been attracting a number of workers in distribution theory of statistics, as well as those in various application fields, including life sciences, social sciences and quality management; see, *e.g.*, Efron and Tibshirani [8]. Among them, Hall [9, 10, 11], Babu and Singh [1, 2, 3], Beran [4], Bickel and Freedman [6], Singh [17], and other authors have discussed the bootstrap with close attention on asymptotic expansions.

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Most important statistics, including non-parametric bootstrap estimates of population parameters, and, excluding those in time series analysis, are functionals in the empirical distribution F^* based on a p-variate random sample

$$\boldsymbol{X} = (\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n) = \begin{pmatrix} x_{11}, & x_{21}, & \ldots, & x_{n1} \\ \vdots & \vdots & & \vdots \\ x_{1p}, & x_{2p}, & \ldots, & x_{np} \end{pmatrix}$$

of size n drawn from an unknown population F, where

$$F^*(\boldsymbol{t}) = F^*\left((t_1, \ldots, t_p)^T\right) = \frac{1}{n}\#\{\boldsymbol{x}_i \mid \forall j = 1, \ldots, p,\, x_{ij} \le t_j;\; i = 1, \ldots n\}.$$

Hence, those statistics have a kind of symmetric property in the sense of invariance though the elements of $\boldsymbol{X}$ are arbitrarily permuted.

Looking at such symmetry, Niki [15], Nakagawa and Niki [12], and Niki, Nakagawa and Hashiguchi [16] have developed a software toolkit for transforming polynomials in sums of power products into (multivariate) augmented monomial symmetric functions, in order to obtain higher (approximate) moments of statistics as asymptotic expansions in $1/\sqrt{n}$. Nakagawa and Niki [13], as well as Nakagawa, Niki and Hashiguchi [14], have clearly demonstrated the role of computer algebra in obtaining asymptotic expansions for the probability integrals of sampling distributions. In this article, discussion is extended to *bootstrap distributions*.

Let $\boldsymbol{Y} = (\boldsymbol{y}_1, \ldots, \boldsymbol{y}_n)$ denote a bootstrap sample of size n resampled from the empirical distribution F^*. The unknown distribution G of the target statistic $T(\boldsymbol{X})$ is approximated with the distribution G^* of a statistic $T(\boldsymbol{Y})$ called “the bootstrap distribution of T” which is computable by using the Monte-Carlo method.

Our aim is to design a set of algorithms for expressing the moments of G^* and the “moments of moments” of G^* in terms of the moments and product moments of F^* and in those of F, respectively. The toolkit due to Nakagawa and Niki [12] coded in REDUCE is rewritten in MATHEMATICA to be extended for th at purpose. Here, existence of the finite moments of F of requisite order is assumed throughout this article.

Derivation of the moments of bootstrap distributions and the *moments of bootstrap moments* for the distributions of the sample correlation coefficient and of the sample regression coefficient, by using the extended version of the symmetric polynomial toolkit, is demonstrated in the last section. Several higher order terms given there are new as far as the authors have known.

2 Definition and preliminaries

Let $\succeq$ denote the reverse lexicographic ordering over the set of p-dimensional vectors of non-negative integers defined by

$$\begin{aligned}&\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_p)^T \succ \boldsymbol{\mu} = (\mu_1, \ldots, \mu_p)^T \\ &\qquad \iff \exists i \in \{1, \ldots, p\}\,(\lambda_1 = \mu_1,\, \ldots,\, \lambda_{i-1} = \mu_{i-1},\, \lambda_i > \mu_i),\end{aligned}$$

where the symbol ^{T} denotes transposition. Then, define the set $\mathcal{F}(p)$ of "p-partitions" as

$$\mathcal{F}(p) = \left\{\boldsymbol{\Lambda} = (\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \ldots,) \mid \boldsymbol{\lambda}_1 \succeq \boldsymbol{\lambda}_2 \succeq \cdots \succeq \mathbf{0};\ \boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \ldots \in \mathbb{N}^p\right\}.$$

The number of non-zero vectors in $\boldsymbol{\Lambda} \in \mathcal{F}(p)$ is written as $\ell(\boldsymbol{\Lambda})$ and is called the length of $\boldsymbol{\Lambda}$. Two p-partitions are identical, if and only if they have the same non-zero parts. The set $\mathcal{F}(p, n)$ of "p-partitions of length not exceeding n" is then written as

$$\mathcal{F}(p, n) = \{\boldsymbol{\Lambda} \mid \boldsymbol{\Lambda} \in \mathcal{F}(p),\ \ell(\boldsymbol{\Lambda}) \le n\}.$$

We also use the notation

$$\boldsymbol{\Lambda} = (\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \ldots) = ({\boldsymbol{\lambda}'_1}^{\pi_1}, {\boldsymbol{\lambda}'_2}^{\pi_2}, \ldots),$$

where $\pi_1, \pi_2, \ldots$ are the multiplicities of the distinct non-zero parts in $\boldsymbol{\Lambda}$:

$$\boldsymbol{\lambda}'_1 \succ \boldsymbol{\lambda}'_2 \succ \cdots \succ \mathbf{0}, \quad \pi_1 + \pi_2 + \cdots = \ell(\boldsymbol{\Lambda}).$$

A p-partition is reduced to an ordinary partition if $p = 1$.

For any $p \times n$ matrix $\boldsymbol{U} = (\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n)$ such that $\boldsymbol{u}_i = (u_{i1}, \ldots, u_{ip})^T \in \mathbb{N}^p$ $(i = 1, \ldots, n)$, we write a monomial in n independent p-dimensional vectors (*i.e.*, in pn independent variables) as

$$\boldsymbol{X}^{\boldsymbol{U}} = \boldsymbol{x}_1^{\boldsymbol{u}_1} \cdots \boldsymbol{x}_n^{\boldsymbol{u}_n} = x_{11}^{u_{11}} \cdots x_{1p}^{u_{1p}} \cdots x_{n1}^{u_{n1}} \cdots x_{np}^{u_{np}}.$$

Then, for any $\boldsymbol{\Lambda} = (\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_n) \in \mathcal{F}(p, n)$, we define the "augmented monomial symmetric polynomial in n of p-dimensional vectors", $\mathcal{A}_{\boldsymbol{\Lambda}}$, by letting

$$\mathcal{A}_{\boldsymbol{\Lambda}} = \mathcal{A}_{\boldsymbol{\Lambda}}(\boldsymbol{X}) = \frac{1}{(n - \ell(\boldsymbol{\Lambda}))!} \sum_{\sigma \in S_n} ({}^{\sigma}\boldsymbol{X})^{\boldsymbol{\Lambda}}, \tag{1}$$

where

$${}^{\sigma}\boldsymbol{X} = {}^{\sigma}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) = (\boldsymbol{x}_{\sigma(1)}, \ldots, \boldsymbol{x}_{\sigma(n)}).$$

Note that $n - \ell(\boldsymbol{\Lambda})$ in (1) is the number of zero vectors in $\boldsymbol{\Lambda}$. We also define the "power sum in n of p-dimensional vectors" $\mathcal{P}_{\boldsymbol{\Lambda}}$ for $\boldsymbol{\Lambda} = ({\boldsymbol{\lambda}'_1}^{\pi_1}, \ldots, {\boldsymbol{\lambda}'_t}^{\pi_t}) \in \mathcal{F}(p)$, provided $\ell(\boldsymbol{\Lambda}) < \infty$, as

$$\mathcal{P}_{\boldsymbol{\Lambda}} = \mathcal{P}_{\boldsymbol{\Lambda}}(\boldsymbol{X}) = \mathcal{P}_{\boldsymbol{\lambda}'_1}^{\pi_1} \cdots \mathcal{P}_{\boldsymbol{\lambda}'_t}^{\pi_t} = \prod_{j=1}^{t} \left(\sum_{i=1}^{n} \boldsymbol{x}_i^{\boldsymbol{\lambda}'_j} \right)^{\pi_j}. \tag{2}$$

The polynomials $\mathcal{A}_{\mathbf{\Lambda}}(\boldsymbol{X})$ and $\mathcal{P}_{\mathbf{\Lambda}}(\boldsymbol{X})$ in p-dimensional vectors are obviously invariant against the action ${}^{\sigma}\boldsymbol{X}$ of any permutation $\sigma \in S_n$ upon $\boldsymbol{X}$ to be called "p-symmetric" in this sense. The sets $\mathcal{A}_{\mathcal{F}(p,n)} = \{\mathcal{A}_{\mathbf{\Lambda}} \mid \mathbf{\Lambda} \in \mathcal{F}(p,n)\}$ and $\mathcal{P}_{\mathcal{F}(p)} = \{\mathcal{P}_{\mathbf{\Lambda}} \mid \mathbf{\Lambda} \in \mathcal{F}(p)\}$ are two bases of the vector space $\{\mathcal{A}_{\mathcal{F}(p,n)}\} \cong \{\mathcal{P}_{\mathcal{F}(p)}\}$ spanned by all the p-symmetric polynomials in n vectors. See Nakagawa and Niki [12], for related topics.

A symbolic algorithm called Algorithm `PtoA` for changing bases from $\mathcal{P}_{\mathcal{F}(p)}$ to $\mathcal{A}_{\mathcal{F}(p,n)}$ has been designed by Nakagawa and Niki [12], with which we can systematically calculate the coefficients $C_{\mathbf{\Lambda M}}$ in

$$\mathcal{P}_{\mathbf{\Lambda}} = \sum_{\mathbf{M} \in \mathcal{F}(p,n)} C_{\mathbf{\Lambda M}}\, \mathcal{A}_{\mathbf{M}}. \tag{3}$$

If $\boldsymbol{X} = (\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n)$ is a p-variate random sample of size n drawn from a population F with the product moments

$$\mu_{\boldsymbol{r}} = \mathrm{E}_F\, \boldsymbol{x}_i^{\boldsymbol{r}} = \mathrm{E}_F\, x_{i1}^{r_1} x_{i2}^{r_2} \cdots x_{ip}^{r_p} \tag{4}$$

of degrees $\boldsymbol{r} = (r_1, r_2, \ldots, r_p)^T \in \mathbb{N}^p$, then $n^{-\ell(\mathbf{\Lambda})}\, \mathcal{P}_{\mathbf{\Lambda}}$ signifies a product of the sample product moments of which degrees are given as $\mathbf{\Lambda}$. Hence, from (3) and the fact that

$$\mathrm{E}_F\, \mathcal{A}_{\mathbf{\Lambda}} = n^{[\ell(\mathbf{\Lambda})]}\, \mu_{\boldsymbol{\lambda}_1'}^{\pi_1} \cdots \mu_{\boldsymbol{\lambda}_t'}^{\pi_t}, \tag{5}$$

where $n^{[k]} = n(n-1)\cdots(n-k+1)$, we can write the expectation of polynomials in sample product moments as polynomials in μ's. By using Algorithm `PtoA`, Nakagawa and Niki [13] have derived an asymptotic expansion of the distribution of sample correlation coefficient from non-normal population.

3 Algorithms

Let $\boldsymbol{X} = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)$ and $\boldsymbol{Y} = (\boldsymbol{y}_1, \ldots, \boldsymbol{y}_n)$ be, as defined in Section 1, a p-variate random sample of size n drawn from a population F, and a bootstrap sample of size n resampled from the empirical distribution F^* composed from $\boldsymbol{X}$.

Our aim is to express the moments of the bootstrap distribution G^* of a statistic $T(\boldsymbol{Y})$ and the *moments of moments* of G^* in terms of the moments and product moments of F^* and in those of F, respectively.

3.1 Expectation of bootstrap product moments

3.1.1 Theoretical background

The following lemma is fundamental in designing symbolic algorithms for the bootstrap (product) moments.

Lemma 1

Let $\boldsymbol{y} \sim F^*$ then, for any $\boldsymbol{\lambda} \in \mathbb{N}^p$, it holds that

$$\mathrm{E}_{F^*}\, \boldsymbol{y}^{\boldsymbol{\lambda}} = \frac{1}{n}\, \mathcal{P}_{\boldsymbol{\lambda}}(\boldsymbol{X}),$$

Proof

$$\mathrm{E}_{F^*}\, \boldsymbol{y}^{\boldsymbol{\lambda}} = \int \boldsymbol{y}^{\boldsymbol{\lambda}} dF^* = \sum_{i=1}^{n} \frac{1}{n}\, \boldsymbol{x}_i^{\boldsymbol{\lambda}} = \frac{1}{n}\, \mathcal{P}_{\boldsymbol{\lambda}}(\boldsymbol{X}),$$

which gives the lemma. ∎

From the preceding lemma, we have the theorem given below corresponding to (5).

Theorem 2

For any $\boldsymbol{\Lambda} \in \mathcal{F}(p, n)$,

$$\mathrm{E}_{F^*}\, \mathcal{A}_{\boldsymbol{\Lambda}}(\boldsymbol{Y}) = \frac{n^{[\ell]}}{n^{\ell}}\, \mathcal{P}_{\boldsymbol{\Lambda}}(\boldsymbol{X}),$$

where $\ell = \ell(\boldsymbol{\Lambda})$.

Proof From Lemma 1, we have $\mathrm{E}_{F^*}\, \boldsymbol{y}_i^{\boldsymbol{\lambda}} = n^{-1}\, \mathcal{P}_{\boldsymbol{\lambda}}$ for any $i \in \{1, \ldots, n\}$ and $\boldsymbol{\lambda} \succ \boldsymbol{0}$. Then the fact that $|S_n| = n!$ gives

$$\begin{aligned}
\mathrm{E}_{F^*}\, \mathcal{A}_{\boldsymbol{\Lambda}}(\boldsymbol{Y}) &= \frac{1}{(n-\ell)!} \sum_{\sigma \in S_n} \mathrm{E}_{F^*} ({}^{\sigma}\boldsymbol{Y})^{\boldsymbol{\Lambda}} = \frac{1}{(n-\ell)!} \sum_{\sigma \in S_n} \prod_{j=1}^{\ell} \mathrm{E}_{F^*}\, \boldsymbol{y}_{\sigma(j)}^{\boldsymbol{\lambda}_j} \\
&= \frac{1}{(n-\ell)!} \sum_{\sigma \in S_n} \prod_{j=1}^{\ell} \frac{1}{n}\, \mathcal{P}_{\boldsymbol{\lambda}_j}(\boldsymbol{X}) = \frac{n!}{(n-\ell)!\, n^{\ell}} \prod_{j=1}^{\ell} \mathcal{P}_{\boldsymbol{\lambda}_j}(\boldsymbol{X}) \\
&= \frac{n^{[\ell]}}{n^{\ell}}\, \mathcal{P}_{\boldsymbol{\Lambda}}(\boldsymbol{X}).
\end{aligned}$$

∎

3.1.2 Algorithm for expectation of bootstrap product moments

The following algorithm furnishes us with the break in making the resulting algorithms for computing the moments or the *approximate* moments of G^*.

Algorithm 1 (Expectation of $\mathcal{P}_{\boldsymbol{\Lambda}}(\boldsymbol{Y})$ with respect to F^*)

Input: $\boldsymbol{\Lambda} \in \mathcal{F}(p)$

Output: $\mathrm{E}_{F^*}\, \mathcal{P}_{\boldsymbol{\Lambda}}(\boldsymbol{Y})$ as a member of the vector space $\{\mathcal{P}_{\mathcal{F}(p,n)}(\boldsymbol{X})\}$

1. Let $w \leftarrow \mathcal{P}_{\boldsymbol{\Lambda}}(\boldsymbol{Y})$ and transform it into the space $\{\mathcal{A}_{\mathcal{F}(p,n)}(\boldsymbol{Y})\}$ by applying Algorithm `PtoA`.
2. Take expectation for each term in w, by using Theorem 2, to give $\overline{w} \leftarrow \mathrm{E}_{F^*}\, w$.
3. Return $\overline{w}$ after simplification.

3.2 Bootstrap moments of statistics

3.2.1 Approximate moments of statistics

It is well known that quite many statistics, including the sample correlation coefficient and the sample regression coefficient, are *smooth* functions of the sample (product) moments, *i.e.*, of the members of $\mathcal{P}_{\mathcal{F}(p)}$. The distributions of those statistics satisfy the Cornish-Fisher assumption and have asymptotic expansions formally constructible from their truncated formal expansions for the moments, namely, *approximate* moments. See Bhattacharya and Ghosh [5], for details.

Such a *smooth* statistic $T(\boldsymbol{X})$ tends to a population parameter θ of which value is usually to be estimated, as n tends to infinity, with asymptotic speed of convergence $\mathrm{E}_F\left(T(\boldsymbol{X})-\theta\right)^2 \sim O(n^{-1})$. Then the semi-standardized random variable W_n composed from $T(\boldsymbol{X})$ as

$$W_n = \sqrt{n}\left(T(\boldsymbol{X})-\theta\right), \tag{6}$$

can be expanded as in the form

$$W_n = W_{n0} + \frac{1}{\sqrt{n}} W_{n1} + \frac{1}{n} W_{n2} + \cdots , \tag{7}$$

where each W_{ni} $(i = 1, 2, \ldots)$ is a polynomial in the members of $\mathcal{P}_{\mathcal{F}(p)}(\boldsymbol{X})$ and independent to n. It is also well known, concerning the k-th moments, that

$$\mathrm{E}_F\, W_n^k \sim \begin{cases} O\left(n^{-\frac{1}{2}}\right), & \text{if } k \text{ is odd;} \\ O(1), & \text{if } k \text{ is even.} \end{cases} \tag{8}$$

The bootstrap analogue W_n^* to (6) is defined as

$$\begin{aligned} W_n^* &= \sqrt{n}\left(T(\boldsymbol{Y})-T(\boldsymbol{X})\right) & (9)\\ &= \sqrt{n}\left(T(\boldsymbol{Y})-\theta\right) - \sqrt{n}\left(T(\boldsymbol{X})-\theta\right) & (10) \end{aligned}$$

and has the corresponding asymptotic properties to (7) and (8).

3.2.2 Algorithms for bootstrap moments of statistics

An algorithm for obtaining the k-th *approximate* moment ${}^s\nu_k$ of W_n^*, about the origin, which satisfies

$$\mathrm{E}_{F^*}\, W_n^{*k} = {}^s\nu_k + o(n^{-\frac{s}{2}})$$

is outlined in the following.

Algorithm 2 (Moment or approximate moment of G^*)

Input: k, s and $W_n^* = \sqrt{n}\,\bigl(T(\boldsymbol{Y}) - T(\boldsymbol{X})\bigr)$.

Output: ${}^s\nu_k = \sum_{\boldsymbol{\Lambda}\in\mathcal{F}(p,n)} c_{\boldsymbol{\Lambda}} \mathcal{P}_{\boldsymbol{\Lambda}}(\boldsymbol{X}) = \mathrm{E}_{F^*}\, {W_n^*}^k + o(n^{-\frac{s}{2}})$.

1. Generate the truncated Taylor expansion for ${W_n^*}^k$ about the origin up to the terms of $O(n^{-\frac{s}{2}})$ and set it into Q. Simplify Q to be a member of $\{\mathcal{P}_{\mathcal{F}(p)}(\boldsymbol{Y})\}$.
2. Compute ${}^s\nu_k \leftarrow \mathrm{E}_{F^*}\, Q$, by applying Algorithm 1 on each $\mathcal{P}_{\boldsymbol{\Lambda}}(\boldsymbol{Y})$ $\bigl(\boldsymbol{\Lambda} \in \mathcal{F}(p)\bigr)$ in Q.
3. Return ${}^s\nu_k$ after simplification in order that it becomes a member of $\{\mathcal{P}_{\mathcal{F}(p,n)}(\boldsymbol{X})\}$.

3.2.3 Algorithms for moments of bootstrap moments

The t-th moment ${}^s\overline{\nu}_{kt} = \mathrm{E}_F\,({}^s\nu_k)^t$ with respect to F can be obtained by using the following algorithm.

Algorithm 3 ("Moment of moment" of G^*)

Input: s, t and ${}^s\nu_k \in \{\mathcal{P}_{\mathcal{F}(p,n)}(\boldsymbol{X})\}$

Output: ${}^s\overline{\nu}_{kt} \sim \mathrm{E}_F\,({}^s\nu_k)^t$.

1. Set $R \leftarrow ({}^s\nu_k)^t$ after expanding the power, where elimination of terms of $o(n^{-\frac{s}{2}})$ is performed "on the fly".
2. Rewrite each $\mathcal{P}_{\boldsymbol{\Lambda}}(\boldsymbol{X})$ $\bigl(\boldsymbol{\Lambda} \in \mathcal{F}(p,n)\bigr)$ in R into a member of $\{\mathcal{A}_{\mathcal{F}(p,n)}(\boldsymbol{X})\}$ by using Algorithm `PtoA`.
3. Take expectation ${}^s\overline{\nu}_{kt} \leftarrow \mathrm{E}_F\, R$, by applying (5) term by term, then return ${}^s\overline{\nu}_{kt}$ after simplification.

3.2.4 Remarks

For cases that $T(\boldsymbol{X}) \in \{\mathcal{P}_{\mathcal{F}(p)}(\boldsymbol{X})\}$, a more simplified algorithm than Algorithm 2 without computing the Taylor expansion is to be applied for the sake of speed and ease of use.

In actual applications, as shown in the next section, it may be more convenient to divide W_n^* into two parts as in (10) and handle each part separately than to process as it is in the form of (9).

4 Examples

For shortness, we use symbols $\mathcal{P}_{\boldsymbol{\Lambda}}$ and $\mathcal{A}_{\boldsymbol{\Lambda}}$ for denoting $\mathcal{P}_{\boldsymbol{\Lambda}}(\boldsymbol{X})$ and $\mathcal{A}_{\boldsymbol{\Lambda}}(\boldsymbol{X})$, respectively, in this section.

4.1 Bootstrap moments of sample correlation coefficient

The sample correlation coefficient r, based on independent bivariate observations

$$\boldsymbol{X} = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) = \big((x_{11}, x_{12})^T, \ldots, (x_{n1}, x_{n2})^T\big)$$

from a non-degenerate population F having finite moments of requisite order and its bootstrap version r^* are defined as

$$r = \frac{s_{12}(\boldsymbol{X})}{\sqrt{s_{11}(\boldsymbol{X}) s_{22}(\boldsymbol{X})}}, \qquad r^* = \frac{s_{12}(\boldsymbol{Y})}{\sqrt{s_{11}(\boldsymbol{Y}) s_{22}(\boldsymbol{Y})}},$$

respectively, where

$$\begin{aligned} s_{11}(\boldsymbol{X}) &= \frac{1}{n}\sum_{i=1}^n x_{i1}^2 - \left(\frac{1}{n}\sum_{i=1}^n x_{i1}\right)^2 = \frac{1}{n}\mathcal{P}_{\left[\begin{smallmatrix}2\\0\end{smallmatrix}\right]} - \frac{1}{n^2}\mathcal{P}_{\left[\begin{smallmatrix}1&1\\0&0\end{smallmatrix}\right]}, \\ s_{22}(\boldsymbol{X}) &= \frac{1}{n}\sum_{i=1}^n x_{i2}^2 - \left(\frac{1}{n}\sum_{i=1}^n x_{i2}\right)^2 = \frac{1}{n}\mathcal{P}_{\left[\begin{smallmatrix}0\\2\end{smallmatrix}\right]} - \frac{1}{n^2}\mathcal{P}_{\left[\begin{smallmatrix}0&0\\1&1\end{smallmatrix}\right]}, \\ s_{12}(\boldsymbol{X}) &= \frac{1}{n}\sum_{i=1}^n x_{i1}x_{i2} - \left(\frac{1}{n}\sum_{i=1}^n x_{i1}\right)\left(\frac{1}{n}\sum_{i=1}^n x_{i2}\right) \\ &= \frac{1}{n}\mathcal{P}_{\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]} - \frac{1}{n^2}\mathcal{P}_{\left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right]}. \end{aligned} \tag{11}$$

We note that the statistic r is the maximum likelihood estimator of the population correlation coefficient of ρ, if F is a bivariate normal distribution.

Our first goal in this example is to obtain

$$\mathrm{E}_F\,\mathrm{E}_{F^*}\big(\sqrt{n}(r^* - r)\big) = \mathrm{E}_F\,\mathrm{E}_{F^*}\big(\sqrt{n}(r^* - \rho)\big) - \mathrm{E}_F\big(\sqrt{n}(r - \rho)\big),$$

where the second term $\mathrm{E}_F\big(\sqrt{n}(r-\rho)\big)$ has been already discussed in bibliographies including Nakagawa and Niki [13]. For expanding the first term as a power series in terms of $1/\sqrt{n}$, we introduce the following three auxiliary variables defined as

$$\begin{aligned} w_{20}^* &= \frac{\sqrt{n}\big(s_{11}(\boldsymbol{Y}) - \kappa_{20}\big)}{\kappa_{20}} \sim O(1), \quad w_{02}^* = \frac{\sqrt{n}\big(s_{22}(\boldsymbol{Y}) - \kappa_{02}\big)}{\kappa_{02}} \sim O(1), \\ w_{11}^* &= \frac{\sqrt{n}\big(s_{12}(\boldsymbol{Y}) - \kappa_{11}\big)}{\kappa_{11}} \sim O(1), \end{aligned} \tag{12}$$

to obtain

$$\begin{aligned}\sqrt{n}(r^*-\rho) \quad &\sim \quad \rho\Bigg\{\left(-\frac{1}{2}\,w^*_{20}+w^*_{11}-\frac{1}{2}\,w^*_{02}\right)\\ &+\frac{1}{\sqrt{n}}\left(\frac{3}{8}\,{w^*_{20}}^2-\frac{1}{2}\,w^*_{20}\,w^*_{11}+\frac{1}{4}\,w^*_{20}\,w^*_{02}-\frac{1}{2}\,w^*_{11}\,w^*_{02}+\frac{3}{8}\,{w^*_{02}}^2\right)\\ &+\frac{1}{n}\left(-\frac{5}{16}\,{w^*_{20}}^3+\frac{3}{8}\,{w^*_{20}}^2\,w^*_{11}-\frac{3}{16}\,{w^*_{20}}^2\,w^*_{02}+\frac{1}{4}\,w^*_{20}\,w^*_{11}\,w^*_{02}\right.\\ &\left.-\frac{3}{16}\,w^*_{20}\,{w^*_{02}}^2+\frac{3}{8}\,{w^*_{02}}^2\,w^*_{11}-\frac{5}{16}\,{w^*_{02}}^3\right)\Bigg\}+O(n^{-\frac{3}{2}}),\end{aligned}$$

where κ_{ij} $(i,j=0,1,2,\ldots)$ denote the (product) cumulants of F.

The expectation of this asymptotic expansion with respect to F^* is given by using ALGORITHM 2 with $p=2$ and $k=1$:

$$\begin{aligned}\mathrm{E}_{F^*}\left(\sqrt{n}(r^*-\rho)\right) \quad &= \quad \frac{1}{\sqrt{n}}\cdot\frac{1}{\sqrt{\kappa_{20}\kappa_{02}}}\left(\frac{-2\,\kappa_{11}}{\kappa_{02}}\,\mathcal{P}_{\left[\begin{smallmatrix}0\\2\end{smallmatrix}\right]}+4\,\mathcal{P}_{\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]}-\frac{2\,\kappa_{11}}{\kappa_{20}}\,\mathcal{P}_{\left[\begin{smallmatrix}2\\0\end{smallmatrix}\right]}\right)\\ &\quad +O(n^{-\frac{3}{2}})\end{aligned}$$

Finally, ALGORITHM 3 and the relations between moments and cumulants yield

$$\begin{aligned}\mathrm{E}_F\,\mathrm{E}_{F^*}\left(\sqrt{n}(r^*-\rho)\right)=\frac{1}{\sqrt{n}}\cdot\frac{1}{\sqrt{\kappa_{20}\kappa_{02}}}&\left(-\kappa_{11}+\frac{3}{4}\,\kappa_{02}^{-2}\kappa_{04}\kappa_{11}+\kappa_{20}^{-1}\kappa_{02}^{-1}\kappa_{11}^3\right.\\ &\left.-\kappa_{02}^{-1}\kappa_{13}+\frac{1}{2}\,\kappa_{20}^{-1}\kappa_{02}^{-1}\kappa_{11}\kappa_{22}-\kappa_{20}^{-1}\kappa_{31}+\frac{3}{4}\,\kappa_{20}^{-2}\kappa_{11}\kappa_{40}\right)+O(n^{-\frac{3}{2}}),\end{aligned}$$

which gives

$$\mathrm{E}_F\,\mathrm{E}_{F^*}\left(\sqrt{n}(r^*-r)\right)=\mathrm{E}_F\left(\sqrt{n}(r-\rho)\right)+O(n^{-\frac{3}{2}})$$

showing that the bootstrap bias correction works well, at least, of $O\left(n^{-1}\right)$.

The expectation of the second moment with respect to F^* and F is expanded in similar fashion as

$$\mathrm{E}_F\,\mathrm{E}_{F^*}\left(n(r^*-r)^2\right)=a_{20}+\frac{a_{22}}{n}+O(n^{-2}), \tag{13}$$

where

$$\begin{aligned}a_{20}=\frac{1}{\kappa_{20}\kappa_{02}}&\left(\kappa_{02}\,\kappa_{20}+\kappa_{22}-\kappa_{11}\,\kappa_{13}\,\kappa_{02}^{-1}-2\,\kappa_{11}^2+\frac{1}{4}\kappa_{04}\,\kappa_{02}^{-2}\,\kappa_{11}^2+\frac{1}{4}\kappa_{40}\,\kappa_{11}^2\,\kappa_{20}^{-2}\right.\\ &\left.-\kappa_{11}\,\kappa_{31}\,\kappa_{20}^{-1}+\frac{1}{2}\kappa_{22}\,\kappa_{02}^{-1}\,\kappa_{11}^2\,\kappa_{20}^{-1}+\kappa_{02}^{-1}\,\kappa_{11}^4\,\kappa_{20}^{-1}\right),\\ a_{22}=\frac{1}{\kappa_{20}\kappa_{02}}&\left(-3\,\kappa_{02}\,\kappa_{20}-\frac{43}{2}\,\kappa_{22}-\frac{69}{8}\,\kappa_{04}\,\kappa_{11}\,\kappa_{13}\,\kappa_{02}^{-3}+20\,\kappa_{03}\,\kappa_{11}\,\kappa_{12}\,\kappa_{02}^{-2}\right.\\ &+\frac{15}{4}\,\kappa_{11}\,\kappa_{15}\,\kappa_{02}^{-2}+\frac{9}{4}\,\kappa_{04}\,\kappa_{22}\,\kappa_{02}^{-2}+\frac{49}{2}\,\kappa_{11}\,\kappa_{13}\,\kappa_{02}^{-1}-\frac{3}{4}\,\kappa_{04}\,\kappa_{20}\,\kappa_{02}^{-1}-7\,\kappa_{03}\,\kappa_{21}\,\kappa_{02}^{-1}\end{aligned}$$

$$
\begin{aligned}
&- 3\,\kappa_{24}\,\kappa_{02}^{-1} + \frac{79}{4}\,\kappa_{11}^{2} - \frac{9}{8}\,\kappa_{06}\,\kappa_{02}^{-3}\,\kappa_{11}^{2} - \frac{53}{8}\,\kappa_{04}\,\kappa_{02}^{-2}\,\kappa_{11}^{2} - 6\,\kappa_{02}^{-3}\,\kappa_{03}^{2}\,\kappa_{11}^{2}\\
&+ \frac{183}{64}\,\kappa_{02}^{-4}\,\kappa_{04}^{2}\,\kappa_{11}^{2} - 9\,\kappa_{02}^{-1}\,\kappa_{12}^{2} + 4\,\kappa_{02}^{-2}\,\kappa_{13}^{2} - \frac{69}{8}\,\kappa_{11}\,\kappa_{31}\,\kappa_{40}\,\kappa_{20}^{-3}\\
&- \frac{9}{8}\,\kappa_{60}\,\kappa_{11}^{2}\,\kappa_{20}^{-3} + \frac{69}{16}\,\kappa_{22}\,\kappa_{40}\,\kappa_{02}^{-1}\,\kappa_{11}^{2}\,\kappa_{20}^{-3} + \frac{69}{8}\,\kappa_{40}\,\kappa_{02}^{-1}\,\kappa_{11}^{4}\,\kappa_{20}^{-3}\\
&+ 20\,\kappa_{11}\,\kappa_{21}\,\kappa_{30}\,\kappa_{20}^{-2} + \frac{9}{4}\,\kappa_{22}\,\kappa_{40}\,\kappa_{20}^{-2} + \frac{15}{4}\,\kappa_{11}\,\kappa_{51}\,\kappa_{20}^{-2} - \frac{29}{4}\,\kappa_{11}\,\kappa_{22}\,\kappa_{31}\,\kappa_{02}^{-1}\,\kappa_{20}^{-2}\\
&- \frac{25}{8}\,\kappa_{11}\,\kappa_{13}\,\kappa_{40}\,\kappa_{02}^{-}1\,\kappa_{20}^{-2} - \frac{53}{8}\,\kappa_{40}\,\kappa_{11}^{2}\,\kappa_{20}^{-2} + \frac{39}{32}\,\kappa_{04}\,\kappa_{40}\,\kappa_{02}^{-2}\,\kappa_{11}^{2}\,\kappa_{20}^{-2}\\
&- 2\,\kappa_{12}\,\kappa_{30}\,\kappa_{02}^{-1}\,\kappa_{11}^{2}\,\kappa_{20}^{-2} - \frac{15}{8}\,\kappa_{42}\,\kappa_{02}^{-1}\,\kappa_{11}^{2}\,\kappa_{20}^{-2} - \frac{59}{2}\,\kappa_{31}\,\kappa_{02}^{-1}\,\kappa_{11}^{3}\,\kappa_{20}^{-2}\\
&+ \frac{55}{4}\,\kappa_{22}\,\kappa_{02}^{-2}\,\kappa_{11}^{4}\,\kappa_{20}^{-2} + \frac{55}{4}\,\kappa_{02}^{-2}\,\kappa_{11}^{6}\,\kappa_{20}^{-2} - 7\,\kappa_{12}\,\kappa_{30}\,\kappa_{20}^{-1} + \frac{49}{2}\,\kappa_{11}\,\kappa_{31}\,\kappa_{20}^{-1}\\
&- \frac{3}{4}\,\kappa_{02}\,\kappa_{40}\,\kappa_{20}^{-1} - 3\,\kappa_{42}\,\kappa_{20}^{-1} - \frac{29}{4}\,\kappa_{11}\,\kappa_{13}\,\kappa_{22}\,\kappa_{02}^{-2}\,\kappa_{20}^{-1} - \frac{25}{8}\,\kappa_{04}\,\kappa_{11}\,\kappa_{31}\,\kappa_{02}^{-2}\,\kappa_{20}^{-1}\\
&+ 23\,\kappa_{11}\,\kappa_{12}\,\kappa_{21}\,\kappa_{02}^{-1}\,\kappa_{20}^{-1} + \kappa_{03}\,\kappa_{11}\,\kappa_{30}\,\kappa_{02}^{-1}\,\kappa_{20}^{-1} + 4\,\kappa_{13}\,\kappa_{31}\,\kappa_{02}^{-1}\,\kappa_{20}^{-1}\\
&+ \frac{9}{2}\,\kappa_{11}\,\kappa_{33}\,\kappa_{02}^{-1}\,\kappa_{20}^{-1} + \frac{69}{16}\,\kappa_{04}\,\kappa_{22}\,\kappa_{02}^{-3}\,\kappa_{11}^{2}\,\kappa_{20}^{-1} - 2\,\kappa_{03}\,\kappa_{21}\,\kappa_{02}^{-2}\,\kappa_{11}^{2}\,\kappa_{20}^{-1}\\
&- \frac{15}{8}\,\kappa_{24}\,\kappa_{02}^{-2}\,\kappa_{11}^{2}\,\kappa_{20}^{-1} + \frac{61}{4}\,\kappa_{22}\,\kappa_{02}^{-1}\,\kappa_{11}^{2}\,\kappa_{20}^{-1} - \frac{59}{2}\,\kappa_{13}\,\kappa_{02}^{-2}\,\kappa_{11}^{3}\,\kappa_{20}^{-1}\\
&+ \frac{69}{8}\,\kappa_{04}\,\kappa_{02}^{-3}\,\kappa_{11}^{4}\,\kappa_{20}^{-1} - \frac{61}{2}\,\kappa_{02}^{-1}\,\kappa_{11}^{4}\,\kappa_{20}^{-1} - 8\,\kappa_{02}^{-2}\,\kappa_{11}^{2}\,{\kappa_{12}}^{2}\,\kappa_{20}^{-1}\\
&- 8\,\kappa_{02}^{-1}\,\kappa_{11}^{2}\,\kappa_{20}^{-2}\,{\kappa_{21}}^{2} - 9\,\kappa_{20}^{-1}\,{\kappa_{21}}^{2} + \frac{55}{16}\,\kappa_{02}^{-2}\,\kappa_{11}^{2}\,\kappa_{20}^{-2}\,{\kappa_{22}}^{2} + \frac{5}{2}\,\kappa_{02}^{-1}\,\kappa_{20}^{-1}\,{\kappa_{22}}^{2}\\
&- 6\,\kappa_{11}^{2}\,\kappa_{20}^{-3}\,{\kappa_{30}}^{2} + 4\,\kappa_{20}^{-2}\,{\kappa_{31}}^{2} + \frac{183}{64}\,\kappa_{11}^{2}\,\kappa_{20}^{-4}\,{\kappa_{40}}^{2}\Bigg). \qquad (14)
\end{aligned}
$$

Nakagawa and Niki [13] have given an expansion for $\mathrm{E}_F\left(n(r-\rho)^2\right)$ as in the form that

$$\mathrm{E}_F\left(n(r-\rho)^2\right) = a'_{20} + \frac{a'_{22}}{n} + O(n^{-2}).$$

Comparing the equation (14) with their results, we have $a_{20} = a'_{20}$ and $a_{22} \neq a'_{22}$.

4.2 Bootstrap moments of sample regression coefficient

Let b and b^* denote the sample regression coefficient and its bootstrap version for the population regression coefficient β, respectively:

$$b = \frac{s_{12}(\boldsymbol{X})}{s_{11}(\boldsymbol{X})}, \qquad b^* = \frac{s_{12}(\boldsymbol{Y})}{s_{11}(\boldsymbol{Y})},$$

where $s_{11}(\boldsymbol{X})$ and $s_{12}(\boldsymbol{X})$ are defined in (11). By using auxiliary variables w^*_{20} and w^*_{11} in (12), we have the asymptotic expansion

$$\sqrt{n}\,(b^*-\beta) \quad \sim \quad \beta\Big\{\left(w^*_{11}-w^*_{20}\right) + \frac{1}{\sqrt{n}}\left(-w^*_{11}w^*_{20}+{w^*_{20}}^2\right) + \frac{1}{n}\left(w^*_{11}{w^*_{20}}^2-{w^*_{20}}^3\right)$$

$$+ \frac{1}{n\sqrt{n}} \left({w^*_{20}}^4 - {w^*_{20}}^3 w^*_{11} \right) \Big\} + O(n^{-2}).$$

The expectations for the first two bootstrap moments of b^* are obtained, by utilizing Algorithm 2 and Algorithm 3, as follows.

$$\begin{aligned} \mathrm{E}_F\, \mathrm{E}_{F^*} \left(\sqrt{n}(b^* - b) \right) &= \frac{c_{11}}{\sqrt{n}} + \frac{c_{13}}{n\sqrt{n}} + O(n^{-\frac{5}{2}}), \\ \mathrm{E}_F\, \mathrm{E}_{F^*} \left(n(b^* - b)^2 \right) &= c_{20} + \frac{c_{22}}{n} + O(n^{-2}), \end{aligned} \tag{15}$$

where

$$c_{11} = \kappa_{20}^{-3} \Big(\kappa_{40}\kappa_{11} - \kappa_{31}\kappa_{20} \Big),$$

$$c_{13} = \kappa_{20}^{-5} \Big(-4\kappa_{60}\kappa_{11}\kappa_{20} + 4\kappa_{51}\kappa_{20}^2 + 9\kappa_{40}^2\kappa_{11} - 9\kappa_{40}\kappa_{31}\kappa_{20} - 17\kappa_{40}\kappa_{11}\kappa_{20}^2 + 17\kappa_{31}\kappa_{20}^3 - 22\kappa_{30}^2\kappa_{11}\kappa_{20} + 22\kappa_{30}\kappa_{21}\kappa_{20}^2 \Big),$$

$$c_{20} = \kappa_{20}^{-4} \Big(-2\,\kappa_{11}\,\kappa_{20}\,\kappa_{31} + \kappa_{40}\,{\kappa_{11}}^2 + \kappa_{22}\,{\kappa_{20}}^2 - {\kappa_{11}}^2\,{\kappa_{20}}^2 + \kappa_{02}\,{\kappa_{20}}^3 \Big),$$

$$\begin{aligned} c_{22} = \kappa_{20}^{-6} \Big(& -26\,\kappa_{11}\,\kappa_{20}\,\kappa_{31}\,\kappa_{40} - 3\,\kappa_{20}\,\kappa_{60}\,{\kappa_{11}}^2 + 28\,\kappa_{11}\,\kappa_{21}\,\kappa_{30}\,{\kappa_{20}}^2 + 4\,\kappa_{22}\,\kappa_{40}\,{\kappa_{20}}^2 \\ & + 6\,\kappa_{11}\,\kappa_{51}\,{\kappa_{20}}^2 - 8\,\kappa_{40}\,{\kappa_{11}}^2\,{\kappa_{20}}^2 - 7\,\kappa_{12}\,\kappa_{30}\,{\kappa_{20}}^3 + 14\,\kappa_{11}\,\kappa_{31}\,{\kappa_{20}}^3 + \kappa_{02}\,\kappa_{40}\,{\kappa_{20}}^3 \\ & - 3\,\kappa_{42}\,{\kappa_{20}}^3 - 7\,\kappa_{22}\,{\kappa_{20}}^4 - 3\,{\kappa_{11}}^2\,{\kappa_{20}}^4 + 3\,\kappa_{02}\,{\kappa_{20}}^5 - 7\,{\kappa_{20}}^3\,{\kappa_{21}}^2 - 14\,\kappa_{20}\,{\kappa_{11}}^2\,{\kappa_{30}}^2 \\ & + 9\,{\kappa_{20}}^2\,{\kappa_{31}}^2 + 13\,{\kappa_{11}}^2\,{\kappa_{40}}^2 \Big). \end{aligned}$$

If we write the results due to Nakagawa and Niki [12] as

$$\begin{aligned} \mathrm{E}_F \left(\sqrt{n}(b - \beta) \right) &= \frac{c'_{11}}{\sqrt{n}} + \frac{c'_{13}}{n\sqrt{n}} + O(n^{-\frac{5}{2}}), \\ \mathrm{E}_F \left(n(b - \beta)^2 \right) &= c'_{20} + \frac{c'_{22}}{n} + O(n^{-2}), \end{aligned} \tag{16}$$

then we have

$$c_{11} = c'_{11}, \quad c_{13} \neq c'_{13}; \quad c_{20} = c'_{20}, \quad c_{22} \neq c'_{22}.$$

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