*Communications of JSSAC (2023)*
*Vol. 5, pp. 1 – 29*

# Algorithms for the crystal structure on K-hives of type $A$

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(Received 14/November/2022 Accepted 28/February/2023)

**Abstract**

A combinatorial object called K-hives realizes the crystal basis of an irreducible highest weight module over the quantized enveloping algebra of type $A$. In this paper, we give a set of algorithms to compute the crystal structure on K-hives. By implementing these algorithms, we created a new package called *khive-crystal* in Python, which incorporates all the functions needed to realize crystal structures. We give some examples of performing this package.

**Keywords:** K-hive, crystal bases, highest weight modules, quantized enveloping algebra, tensor product decomposition

## 1 Introduction

For a symmetrizable Kac-Moody algebra $\mathfrak{g}$, the quantized enveloping algebra $U_q(\mathfrak{g})$ is determined with an indeterminate $q$, see [2, 4]. Certain modules over $U_q(\mathfrak{g})$ have crystal bases, which can be viewed as its basis at $q = 0$, and it enables us to understand the representation theory of $U_q(\mathfrak{g})$ from combinatorics. For example, the irreducible highest weight modules over the quantized enveloping algebra of a simple Lie algebra have a crystal basis and are realized by Young tableaux [7]. Then the action of $U_q(\mathfrak{g})$ on the highest weight modules can be computed by Young tableaux combinatorics. It also means that other problems in representation theory, such as the tensor product decomposition, can be approached by the combinatorics.

In a previous study, we gave a crystal structure on a set of K-hives and showed that the crystal of K-hives is isomorphic to the crystal basis of an irreducible highest weight module over a quantized enveloping algebra of type $A$. K-hive is a labeling of vertices of an equilateral triangular graph introduced in [8, 12, 11]. K-hives have correspondence with semistandard Young tableaux or Gelfand-Tsetlin patterns, and then, for example, they can be applied to compute (Stretched) Kostka coefficients. Also, there is another special kind of hive called LR-hives, which corresponds to Littlewood-Richardson tableaux and has application to Littlewood-Richardson coefficients. For example, in [16], the symmetry of the Littlewood-Richardson coefficients is proved

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by LR-hives. Therefore, hives give a new perspective on problem solving in combinatorial representation theory.

In this paper, we give a set of algorithms for computing the crystal structure given in [14] on K-hives, and some examples of executing these algorithms using the first author's original system. The system is implemented as a Python package named *khive-crystal*. The source code is available in [13].

This paper is organized as follows. In Section 2, we review basic notions and notations of quantized enveloping algebras, crystals, and K-hives. In Section 3, we give a set of algorithms for the crystal structure on a set of K-hives using two approaches. One approach can be obtained by considering a set of K-hives determined by a dominant weight as a subset of a tensor product of sets of K-hives determined by fundamental weights. The other approach is based on an combinatorial description of the crystal structure on K-hives. In Section 5, we give concluding remarks.

# 2 Preliminaries

## 2.1 Quantized Enveloping Algebras

In this subsection, we review the definition of quantized enveloping algebras of type $A$, see [3] for more details.

Let $\mathfrak{sl}_n$ be the Lie algebra of type $A_{n-1}$ over $\mathbb{C}$ with Cartan subalgebra $\mathfrak{h}$ consisting of traceless diagonal matrices. Let $I = \{1, 2, \ldots, n-1\}$ be an index set. Let $A = (a_{ij})_{i,j\in I}$ be the Cartan matrix of type $A_{n-1}$. For $i \in I$, define the liner map $\epsilon_i \colon \mathfrak{h} \to \mathbb{C}$ by $\epsilon_i(h) = \lambda_i$, where $h = \mathrm{diag}(\lambda_j \mid j \in I) \in \mathfrak{h}$. For $i \in I$, set $\alpha_i = \epsilon_i - \epsilon_{i+1}$. Let $\Pi = \{\alpha_i\}_{i\in I} \subset \mathfrak{h}^*$ be simple roots and $\Pi^\vee = \{h_i\}_{i\in I} \subset \mathfrak{h}$ be simple coroots. Let $\Delta$ be the root system of $\mathfrak{sl}_n$. Set $\Delta^+ = \Delta \cap \sum_{i\in I} \mathbb{Z}_{\geq 0}\alpha_i$ and $\Delta^- = \Delta - \Delta^+$. For all $i \in I$, let $\Lambda_i = \epsilon_1 + \epsilon_2 + \cdots + \epsilon_i \in \mathfrak{h}^*$ be an $i$-th fundamental weight. Set $P = \bigoplus_{i\in I} \mathbb{Z}\Lambda_i$, $P^+ = \bigoplus_{i\in I} \mathbb{Z}_{\geq 0}\Lambda_i$, and $P^\vee = \bigoplus_{i\in I} \mathbb{Z}h_i$. We call $P$ the weight lattice, $P^+$ the set of dominant integral weights, and $P^\vee$ the dual weight lattice, respectively. Using this notation, the Cartan datum for $\mathfrak{sl}_n$ is defined as $(A, \Pi, \Pi^\vee, P, P^\vee)$.

Let $q$ be an indeterminate. Let $U_q(\mathfrak{sl}_n)$ be the quantized enveloping algebra over $\mathbb{Q}(q)$ associated with the Cartan datum $(A, \Pi, \Pi^\vee, P, P^\vee)$. Let $V(\lambda)$ be the irreducible highest weight module of weight $\lambda \in P^+$ with the highest weight vector $v_\lambda$ over $U_q(\mathfrak{sl}_n)$.

## 2.2 Crystals

In this subsection, we review the notion of crystals, see [3, 5, 6] for more details.

**Definition 1**

*A* ***crystal*** *associated with Cartan datum* $(A, \Pi, \Pi^\vee, P, P^\vee)$ *is a set* $B$ *together with the maps* $\mathrm{wt}\colon B \to P$, $e_i, f_i \colon B \to B \cup \{0\}$, *and* $\varepsilon_i, \varphi_i \colon B \to \mathbb{Z} \cup \{-\infty\}$ $(i \in I)$ *satisfying the following properties.*

1. $\varphi_i(b) = \varepsilon_i(b) + \mathrm{wt}(b)(h_i)$ *for* $i \in I$,
2. $\mathrm{wt}(e_ib) = \mathrm{wt}(b) + \alpha_i$ *if* $e_ib \in B$,
3. $\mathrm{wt}(f_ib) = \mathrm{wt}(b) - \alpha_i$ *if* $e_ib \in B$,
4. $\varepsilon_i(e_ib) = \varepsilon_i(b) - 1$, $\varphi_i(e_ib) = \varphi_i(b) + 1$ *if* $e_ib \in B$,
5. $\varepsilon_i(f_ib) = \varepsilon_i(b) + 1$, $\varphi_i(f_ib) = \varphi_i(b) - 1$ *if* $f_ib \in B$,

6. $f_i b = b'$ if and only if $b = e_i b'$ for $b, b' \in B$, $i \in I$,

7. if $\varphi_i(b) = -\infty$, then $e_i b = f_i b = 0$.

Since $(A, \Pi, \Pi^\vee, P, P^\vee)$ is the Cartan datum of type $A_{n-1}$, a crystal associated with $(A, \Pi, \Pi^\vee, P, P^\vee)$ is also called a $U_q(\mathfrak{sl}_n)$-crystal.

A $U_q(\mathfrak{sl}_n)$-crystal can be thought of as a colored-oriented graph in the following manner.

**Definition 2**
*Let $B$ be a $U_q(\mathfrak{sl}_n)$-crystal. A* ***crystal graph*** *of $B$ is an $I$-colored oriented graph whose vertices are elements of $B$ and the arrows are written as $b \xrightarrow{i} b'$ when $f_i b = b'$ for $i \in I$ and $b, b' \in B$.*

The tensor product of crystals is defined as follows.

**Definition 3**
*Let $B_1$ and $B_2$ be crystals. The tensor product $B_1 \otimes B_2$ of $B_1$ and $B_2$ is defined to be the set $B_1 \times B_2$ whose crystal structure is defined by*

1. $\mathrm{wt}(b_1 \otimes b_2) = \mathrm{wt}(b_1) + \mathrm{wt}(b_2)$,

2. $\varepsilon_i(b_1 \otimes b_2) = \max(\varepsilon_i(b_1), \varepsilon_i(b_2) - \mathrm{wt}(b_1)(h_i))$,

3. $\varphi(b_1 \otimes b_2) = \max(\varphi(b_2), \varphi(b_1) + \mathrm{wt}(b_2)(h_i))$,

4. $e_i(b_1 \otimes b_2) = \begin{cases} e_i b_1 \otimes b_2 & \varphi_i(b_1) \geq \varepsilon_i(b_2), \\ b_1 \otimes e_i b_2 & \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases}$

5. $f_i(b_1 \otimes b_2) = \begin{cases} f_i b_1 \otimes b_2 & \varphi_i(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes f_i b_2 & \varphi_i(b_1) \leq \varepsilon_i(b_2). \end{cases}$

In general, we have the following proposition([7, Proposition 2.1.1]).

**Proposition 4**
*For $j \in \{1, \ldots, N\}$, let $B_j$ be a $U_q(\mathfrak{sl}_n)$-crystal. Fix $i \in I$. Take $b_j \in B_j$ ($j = 1, \ldots, N$), and we set*

$$a_k = \sum_{1 \leq j < k} \Big(\varphi_i(b_j) - \varepsilon_i(b_{j+1})\Big) \quad 1 \leq k \leq N.$$

*In particular, we set $a_1 = 0$. Then we have*

1. $\varepsilon_i(b_1 \otimes \cdots \otimes b_N) = \max\left\{\sum_{1 \leq j \leq k} \varepsilon_i(b_j) - \sum_{1 \leq j < k} \varphi_i(b_j) \mid 1 \leq k \leq N\right\}$,

2. $\varphi_i(b_1 \otimes \cdots \otimes b_N) = \max\left\{\varphi_i(b_N) + \sum_{k \leq j < N} \Big(\varphi_i(b_j) - \varepsilon_i(b_{j+1})\Big) \mid 1 \leq k \leq N\right\}$,

3. *If $k$ is the largest element such that $a_k = \min\{a_j \mid 1 \leq j \leq N\}$ then, we have*

$$f_i(b_1 \otimes \cdots \otimes b_N) = b_1 \otimes \cdots \otimes b_{k-1} \otimes f_i b_k \otimes b_{k+1} \otimes \cdots \otimes b_N,$$

4. *If $k$ is the smallest element such that $a_k = \min\{a_j \mid 1 \leq j \leq N\}$ then, we have*

$$e_i(b_1 \otimes \cdots \otimes b_N) = b_1 \otimes \cdots \otimes b_{k-1} \otimes e_i b_k \otimes b_{k+1} \otimes \cdots \otimes b_N.$$

An isomorphism of crystals is defined as a bijection preserving crystal structure. Later we will also construct a crystal embedding as defined in the following.

**Definition 5**
*Let $B_1, B_2$ be $U_q(\mathfrak{sl}_n)$-crystals. A* ***crystal morphism*** $\Psi\colon B_1 \to B_2$ *is a map* $\Psi\colon B_1 \cup \{0\} \to B_2 \cup \{0\}$ *satisfying*

1. $\mathrm{wt}(\Psi(b)) = \mathrm{wt}(b), \varepsilon_i(\Psi(b)) = \varepsilon_i(b), \varphi_i(\Psi(b)) = \varphi_i(b)$ *if* $b \in B_1, \Psi(b) \in B_2$,
2. $f_i\Psi(b) = \Psi(f_i b), e_i\Psi(b) = \Psi(e_i b)$ *if* $\Psi(b), \Psi(e_i b), \Psi(f_i b) \in B_2$ *for* $b \in B_1$,
3. $\Psi(0) = 0$.

*A morphism* $\Psi\colon B_1 \to B_2$ *is called an* ***embedding*** *if* $\Psi$ *induces an injection* $B_1 \cup \{0\} \to B_2 \cup \{0\}$. *A morphism* $\Psi\colon B_1 \to B_2$ *is called an* ***isomorphism*** *if* $\Psi$ *induces a bijection* $B_1 \cup \{0\} \to B_2 \cup \{0\}$. *We write* $B_1 \cong B_2$ *if there exists an isomorphism* $\Psi\colon B_1 \to B_2$.

## 2.3 K-hives

Hives are introduced by T.Tao and A.Knutson [12, 11] as the labeling of the vertices of an equilateral triangular graph. There are three forms of hives, one of which, the upright gradient representation, is used in this paper. See [16] for more details. In this paper, we use K-hives, which are a special kind of hives [8].

Let $n \in \mathbb{Z}_{\geq 0}$ and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{Z}^n_{\geq 0}$. $\lambda$ is called a **composition** of $m \in \mathbb{Z}_{\geq 0}$ if $\lambda_1 + \cdots + \lambda_n = m$. A composition $\lambda$ is called a **partition** of $m$ if $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$. If $\lambda$ is a partition of $m$ such that $\lambda_i = k$ for $1 \leq i \leq l \leq n$ and $\lambda_i = 0$ for $l < i \leq n$, then we write $\lambda$ as $(k^m)$. In particular, we simply write $(0^n)$ as 0 if there is no fear of confusion. In addition, $\ell(\lambda)$ denotes the length of $\lambda$.

For $\lambda \in P^+$, there exists a partition $\tilde{\lambda}$ such that $\tilde{\lambda}_1\epsilon_1 + \tilde{\lambda}_2\epsilon_2 + \cdots + \tilde{\lambda}_n\epsilon_n = \lambda$. Similarly, for $\mu \in P$, there exists a composition $\tilde{\mu}$ such that $\tilde{\mu}_1\epsilon_1 + \tilde{\mu}_2\epsilon_2 + \cdots + \tilde{\mu}_n\epsilon_n = \mu$. Note that a composition $(\tilde{\mu}_1 + k, \ldots, \tilde{\mu}_n + k)$ also represents $\mu \in P$ since $\epsilon_1 + \cdots + \epsilon_n = 0$.

Let $\xi \in P$ be a weight of $V(\lambda)$. Then $\xi$ is written as $\lambda - \sum_{i \in I} k_i\alpha_i \in P\,(k_i \in \mathbb{Z})$. For $\xi$, there exists a composition $\tilde{\xi}$ such that $\tilde{\xi}_1\epsilon_1 + \tilde{\xi}_2\epsilon_2 + \cdots + \tilde{\xi}_n\epsilon_n = \xi$ and $\sum_{k=1}^n \tilde{\xi}_k = \sum_{k=1}^n \tilde{\lambda}_k$.

In the following, a partition (resp. composition) $\tilde{\lambda}$ representing a dominant weight (resp. an integral weight) $\lambda$ is also denoted by $\lambda$ by abuse of notation.

**Definition 6**
*Let* $\alpha = (\alpha_1, \ldots, \alpha_n), \beta = (\beta_1, \ldots, \beta_n), \gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{Z}^n$. *Let* $(U_{ij})_{1 \leq i < j \leq n} \in \mathbb{Z}^{n(n-1)/2}$. *An* ***integer hive*** *of size n in upright gradient representation ([16]) is a tuple* $(\alpha, \beta, \gamma, (U_{ij})_{1 \leq i < j \leq n})$ *that satisfies*

$$\beta_k = (\gamma_k + \sum_{i=1}^{k-1} U_{ik}) + (\alpha_k - \sum_{j=k+1}^{n} U_{kj}). \tag{1}$$

**Remark 7**
*In [12, 11, 16], the term hive refers to a hive with additional inequality conditions called the rhombus inequalities. We rather follow the terminology of [8, 9, 10].*

An integer hive in upright gradient representation is illustrated as the labeling of an equilateral triangular graph with boundary edge labels and rhombi as shown in Fig. 1.

Set $[n] = \{1, 2, \ldots, n\}$. In the following, for $i \in [n]$, set

$$U_{ii} = \beta_i - \sum_{k=1}^{i-1} U_{ki} \tag{2}$$

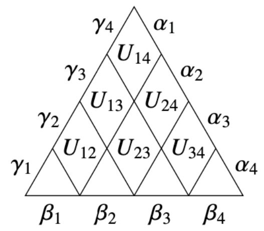



Fig. 1: Integer hive graph of size 4

and $U_{ij} = 0$ if $i > j$ or $j > n$ or $i < 1$. Also, for simplicity, we will write $(U_{ij})_{1 \le i < j \le n}$ as $(U_{ij})_{i<j}$.

In this paper, we consider a kind of integer hive called a K-hive.

**Definition 8**
*Let $m, n \in \mathbb{Z}_{\ge 0}$. Let $\alpha, \beta, \gamma \in \mathbb{Z}^n_{\ge 0}$. For $1 \le i < j \le n$, set $L_{ij} = \sum_{k=1}^{j-1} U_{ik} - \sum_{k=1}^{j} U_{i+1,k}$. Then an integer hive in upright gradient representation $H = (\alpha, \beta, \gamma, (U_{ij})_{i<j})$ is called a **K-hive** if the following conditions are satisfied*

1. *$\alpha$ is a partition of $m$,*
2. *$\beta$ is a composition of $m$,*
3. *$\gamma = (0^n)$,*
4. *$U_{ij} \ge 0$ for $1 \le i < j \le n$,*
5. *$L_{ij} \ge 0$ for $1 \le i < j \le n$,*
6. *$\beta_i \ge \sum_{k=1}^{i-1} U_{ki}$ for $i \in [n]$.*

*Let*

$$\mathcal{H}^{(n)}(\alpha, \beta, 0) = \{H = (\alpha, \beta, 0, (U_{ij})_{i<j}) \mid H \text{ is a K-hive}\}.$$

*Set*

$$\mathbb{H}(\alpha) = \bigcup_{\beta} \mathcal{H}^{(n)}(\alpha, \beta, 0)$$

*where the union runs through all compositions of $m$.*

**Remark 9**
*For $H = (\alpha, \beta, 0, (U_{ij})_{i<j}) \in \mathcal{H}^{(n)}(\alpha, \beta, 0)$, we have*

$$\begin{aligned} \sum_{k=1}^{n} \beta_k &= \sum_{k=1}^{n} (\sum_{i=1}^{k-1} U_{ik} + \alpha_k - \sum_{j=k+1}^{n} U_{kj}) \\ &= \sum_{k=1}^{n} \alpha_k. \end{aligned}$$

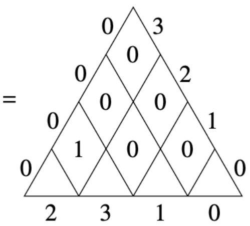
*Thus, if $\sum_{i=1}^{n} \alpha_i \ne \sum_{i=1}^{n} \beta_i$, we have $\mathcal{H}^{(n)}(\alpha, \beta, 0) = \emptyset$.*

**Remark 10**
*Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ be a partition of $m \in \mathbb{Z}_{\geq 0}$. Let $l \in \mathbb{Z}_{\geq 0}$. Set $\alpha' = (\alpha_i + l)_i$. We know that $\alpha$ and $\alpha'$ represent the same dominant weight. We also have that $\mathbb{H}(\alpha) \cong \mathbb{H}(\alpha')$ as a set. The bijection from $\mathbb{H}(\alpha)$ to $\mathbb{H}(\alpha')$ is given by the map which maps $(\alpha, \beta, 0, (U_{ij})_{i<j})$ to $(\alpha', \beta', 0, (U'_{ij})_{i<j})$, where $\beta' = (\beta_i + l)_i$ and $(V_{ij})_{i<j} = (U_{ij})_{i<j}$. Note that $V_{ii} = U_{ii} + l$ holds for $i = 1, 2, \dots, n-1$.*

**Remark 11**
*Let $H \in \mathcal{H}^{(n)}(\alpha, \beta, 0) \subset \mathbb{H}(\alpha)$. In this case, we have $U_{ii} = \alpha_i - \sum_{j=i+1}^{n} U_{ij}$ by Definition 6 (1)(2). Also, we have $U_{ij} = 0$ for $j \in [n]$ if $\alpha_i = 0$ since $U_{kl} \geq 0$ for $1 \leq k \leq l \leq n$.*

**Example 12**
*Let $n = 4$, $\lambda = (3, 2, 1, 0)$ and $\mu = (2, 3, 1, 0)$. We have an example of $H \in \mathcal{H}^{(4)}(\lambda, \mu, 0) \subset \mathbb{H}(\lambda)$ as shown in Fig. 2.*

$H =$




$\in \mathcal{H}^{(4)}(\lambda, \mu, 0) \subset \mathbb{H}(\lambda).$

Fig. 2: An example of a K-hive

**Remark 13**
*Let $\lambda \in P^+$ and let $H = (\lambda, \mu, 0, (U_{ij})_{i<j}) \in \mathbb{H}(\lambda)$. Let $T$ be a Young tableau of shape $\lambda$, weight $\mu$, and let $U_{ij}$ be the number of $j$ in $i$-th row of $T$. Then, the map that sends $H$ to $T$ is a bijection from $\mathbb{H}(\lambda)$ to the set of semistandard tableaux of shape $\lambda$ (cf. [9]).*

## 2.4 Crystal Structure on K-hives

In this subsection, we review the crystal structure on $\mathbb{H}(\lambda)$ for $\lambda \in P^+$ according to [14]. There are two ways to introduce the crystal structure on $\mathbb{H}(\lambda)$. One way is realized by regarding $\mathbb{H}(\lambda)$ as a subset of a tensor product of crystals of the form $\mathbb{H}(\Lambda_k)$. Another way is realized by considering a combinatorial description of the crystal structure.

The following is a technical lemma.

**Lemma 14 ([14])**
*Let $\nu \in I$ and $H = (\Lambda_\nu, \mu, 0, (U_{ij})_{i<j}) \in \mathbb{H}(\Lambda_\nu)$.*

1. *For all $i \in \{1, 2, \dots, \nu\}$, there exists a unique $j \in \{i, i+1, \dots, n\}$ such that $U_{ij} = 1$.*
2. *Fix $j \in I$. If there exists $i, i' \in \{1, 2, \dots, j\}$ such that $U_{ij}, U_{i'j} > 0$, then $i = i'$ holds.*

We first define the crystal structure on $\mathbb{H}(\Lambda_k)$ for $k \in I$.

**Definition 15 ([14])**
*Let $\nu \in I$. The maps* $\mathrm{wt}\colon \mathbb{H}(\Lambda_\nu) \to P$, $e_i, f_i\colon \mathbb{H}(\Lambda_\nu) \to \mathbb{H}(\Lambda_\nu) \cup \{0\}$ *and* $\varepsilon_i, \varphi_i\colon \mathbb{H}(\Lambda_\nu) \to \mathbb{Z}_{\geq 0}$ $(i \in I)$ *are defined in the following manner. Let* $H = (\Lambda_\nu, \mu, 0, (U_{ij})_{i<j}) \in \mathbb{H}(\Lambda_\nu)$.

1. $\mathrm{wt}(H) := \sum_{k=1}^{n-1}(\mu_k - \mu_{k+1})\Lambda_k \in P$,
2. $\varepsilon_i(H) := \max(\mu_{i+1} - \mu_i, 0)$,
3. $\varphi_i(H) := \max(\mu_i - \mu_{i+1}, 0)$,
4. *Set* $\mu' = \sum_{k=1}^{n} \mu'_k \epsilon_k \in P$ *where* $\mu'_i = \mu_i + 1$, $\mu'_{i+1} = \mu_{i+1} - 1$, *and* $\mu'_k = \mu_k$ *for* $k \neq i, i+1$. *Set* $U'_{k_0,i} = U_{k_0,i} + 1$, $U'_{k_0,i+1} = U_{k_0,i+1} - 1$ *if there exists* $k_0 \in \{1, 2, \ldots, i+1\}$ *such that* $U_{k_0,i+1} > 0$. *Set* $U'_{kl} = U_{kl}$ *if* $k \neq k_0$ *and* $l \neq i, i+1$. *Then, for* $i \in I$, $e_i\colon \mathbb{H}(\Lambda_\nu) \to \mathbb{H}(\Lambda_\nu) \cup \{0\}$ *is defined as follows:*

$$e_i H = \begin{cases} (\Lambda_\nu, \mu', 0, (U'_{kl})_{k<l}) & \varepsilon_i(H) > 0, \\ 0 & \varepsilon_i(H) = 0, \end{cases}$$

5. *Set* $\mu' = \sum_{k=1}^{n} \mu'_k \epsilon_k \in P$ *where* $\mu'_i = \mu_i - 1$, $\mu'_{i+1} = \mu_{i+1} + 1$, *and* $\mu'_k = \mu_k$ *for* $k \neq i, i+1$. *Set* $U'_{k_0,i} = U_{k_0,i} - 1$, $U'_{k_0,i+1} = U_{k_0,i+1} + 1$ *if there exists* $k_0 \in \{1, 2, \ldots, i\}$ *such that* $U_{k_0,i} > 0$. *Set* $U'_{kl} = U_{kl}$ *if* $k \neq k_0$ *and* $l \neq i, i+1$. $f_i\colon \mathbb{H}(\Lambda_\nu) \to \mathbb{H}(\Lambda_\nu) \cup \{0\} (i \in I)$ *is defined as follows:*

$$f_i H = \begin{cases} (\Lambda_\nu, \mu', 0, (U'_{kl})_{k<l}) & \varphi_i(H) > 0, \\ 0 & \varphi_i(H) = 0. \end{cases}$$

**Remark 16 ([14])**
*It follows from Definition 6 (1) that* $\mu_i \in \{0, 1\}$ *for all* $i \in [n]$ *since* $\Lambda_\nu$ *corresponds to* $(1^k)$. *Thus, we have* $\varphi_i(H), \varepsilon_i(H) \in \{0, 1\}$. *Moreover, the following holds.*

$$\varphi_i(H) = \begin{cases} 1 & f_i H \neq 0, \\ 0 & f_i H = 0. \end{cases}$$

$$\varepsilon_i(H) = \begin{cases} 1 & e_i H \neq 0, \\ 0 & e_i H = 0. \end{cases}$$

**Proposition 17 ([14])**
*Let* $\nu \in I$. *Then* $\mathbb{H}(\Lambda_\nu)$ *is a* $U_q(\mathfrak{sl}_n)$*-crystal together with the maps* wt, $e_i$, $f_i$, $\varphi_i$, $\varepsilon_i$ *in Definition 15.*

By the map $\Psi$ defined in the following, we regard $\mathbb{H}(\lambda)$ as a subset of a tensor product of crystals of the form $\mathbb{H}(\Lambda_k)$. Then, to define $\Psi$, we first define a map $\Psi_\lambda$.

**Definition 18 ([14])**
*Let* $\lambda = \sum_{i \in I} m_i \Lambda_i \in P^+$. *Set* $N = \sum_{i \in I} m_i$. *Let* $l_N = \max\{i \in I \mid m_i \neq 0\}$. *For* $H = (\lambda, \mu, 0, (U_{ij})_{i<j}) \in \mathbb{H}(\lambda)$, $H_N = (\Lambda_{l_N}, \mu^{(N)}, 0, (U^{(N)}_{ij})_{i<j})$ *is defined by*

$$U^{(N)}_{ij} = \begin{cases} 1 & \text{if } j = \min\{j \in [n] \mid U_{ij} > 0\}, \\ 0 & \text{otherwise}, \end{cases}$$

$$\mu^{(N)}_k = \begin{cases} 1 & \text{if there exists } j \in [n] \text{ such that } U^{(N)}_{kj} > 0, \\ 0 & \text{otherwise} \end{cases}$$

*For $H$ and $H_N$, $H^{(N-1)} = (\lambda^{(N-1)}, \xi^{(N-1)}, 0, (V_{ij}^{(N-1)})_{i<j})$ is defined by $\lambda^{(N-1)} = \lambda - \Lambda_{l_N}$, $\xi^{(N-1)} = \mu - \mu^{(N)}$, and $V_{ij}^{(N-1)} = U_{ij} - U_{ij}^{(N)}$ $(1 \le i < j \le n)$.*

**Lemma 19 ([14])**
*Let $\lambda = \sum_{i\in I} m_i\Lambda_i \in P^+$. Set $N = \sum_{i\in I} m_i$. Let $H \in \mathbb{H}(\lambda)$. Let $H_N$ and $H^{(N-1)}$ in Definition 18. Then, $H_N \in \mathbb{H}(\Lambda_{l_N})$ and $H^{(N-1)} \in \mathbb{H}(\lambda^{(N-1)})$ hold.*

**Definition 20 ([14])**
*Let $\lambda = \sum_{i\in I} m_i\Lambda_i \in P^+$. Set $N = \sum_{i\in I} m_i$. For each $H \in \mathbb{H}(\lambda)$, take $H_N \in \mathbb{H}(\Lambda_{l_N})$ and $H^{(N-1)} \in \mathbb{H}(\lambda^{(N-1)})$ as in Definition 18. Then define the map $\Psi_\lambda : \mathbb{H}(\lambda) \to \mathbb{H}(\lambda^{(N-1)}) \times \mathbb{H}(\Lambda_{l_N})$ by $\Psi_\lambda(H) = H^{(N-1)} \times H_N$.*

**Lemma 21 ([14])**
*The map $\Psi_\lambda$ is an injection.*

By applying $\Psi_\lambda$ repeatedly, we have an injection from $\mathbb{H}(\lambda)$ to $\bigotimes_k \mathbb{H}(\Lambda_k)$.

**Proposition 22 ([14])**
*Let $\lambda = \sum_{i\in I} m_i\Lambda_i \in P^+$. Then there exists an injection*

$$\Psi\colon \mathbb{H}(\lambda) \to \bigotimes_{i\in I} \mathbb{H}(\Lambda_i)^{\otimes m_i}.$$

To define the crystal structure on $\mathbb{H}(\lambda)$ for $\lambda \in P^+$ so that $\Psi$ is a crystal morphism, we need to show that an image of $\Psi$ is stable under the action of $e_i, f_i$ $(i \in I)$. To show this, we start by examining an image of $\Psi$.

**Lemma 23**
*Let $\lambda = \sum_{i\in I} m_i\Lambda_i \in P^+$. Set $N = \sum_{i\in I} m_i$. Let $H \in \mathbb{H}(\lambda)$. Let $\Psi(H) = H_1 \otimes \cdots \otimes H_N$, where $H_k = (\Lambda_{l_k}, \mu^{(k)}, 0, (U_{ij}^{(k)})_{i<j})$ $(k = 1, \ldots, N)$. For $k \in \{1, \ldots, N\}$ and $i \in [n]$, if there exists $j \in [n]$ such that $U_{i,j}^{(k)} > 0$, then set $j_{i,k}$ to its $j$, otherwise set $j_{i,k}$ to $0$. Suppose that $j_{i,k} > 0$ for some $k \in \{1, \ldots, N\}$ and $i \in [n]$. Then we have $j_{i,k'} \ge j_{i,k}$ if $k \ge k'$.*

Proof  Set $H^{(N)} = H$ and $\lambda^{(N)} = \lambda$. By Definition 18, for $m = 1, 2, \ldots, N$ there exists $H_m \in \mathbb{H}(\Lambda_{l_m})$ and $H^{(m-1)} \in \mathbb{H}(\lambda^{(m-1)})$ such that

$$\Psi_{\lambda^{(m)}}(H^{(m)}) = H^{(m-1)} \otimes H_m.$$

For $m = 1, 2, \ldots, N$, let $H^{(m)} = (\lambda^{(m)}, \xi^{(m)}, 0, (V_{ij}^{(m)})_{i<j})$. Fix $k \in \{1, 2, \ldots, N\}$. It follows from the definition of $\Psi$ and $\Psi_\lambda$ $(\lambda \in P^+)$ that

$$V_{ij}^{(k)} = U_{ij}^{(1)} + \cdots + U_{ij}^{(k)} \quad (1 \le i < j \le n).$$

Then, by the definition of $\Psi_{\lambda^{(k)}}$,

$$U_{ij}^{(k)} = \begin{cases} 1 & j = \min\{j \in [n] \mid V_{ij}^{(k)} > 0\}, \\ 0 & \text{else}. \end{cases}$$

This means that for $1 \le k' \le k \le N$

$$\begin{aligned} j_{i,k} &= \min\{j \in [n] \mid U_{ij}^{(1)} + \cdots + U_{ij}^{(k)} > 0\} \\ &\le \min\{j \in [n] \mid U_{ij}^{(1)} + \cdots + U_{ij}^{(k')} > 0\} \\ &= j_{i,k'}. \end{aligned}$$

∎

**Remark 24**
*It follows from Lemma 23 that*

$$\begin{aligned} j_{i,k} &= \min\{j \in [n] \mid U_{ij}^{(l)} > 0, l = 1, \ldots, k\} \\ &= \max\{j \in [n] \mid U_{ij}^{(l)} > 0, l = k, \ldots, N\}. \end{aligned}$$

**Proposition 25**
*Let* $\lambda = \sum_{i\in I} m_i\Lambda_i = \sum_{i\in I} \lambda_i\epsilon_i \in P^+$. *Set* $N = \sum_{i\in I} m_i$. *Then,*

$$\Psi(\mathbb{H}(\lambda)) = \{H_1 \otimes \cdots \otimes H_N \in \bigotimes_{k=1}^{N} \mathbb{H}(\Lambda_{l_k}) \mid j_{i,\lambda_{N+1-i}} \geq j_{i,\lambda_{N+1-i}+1} \geq \cdots \geq j_{i,N} \text{ for all } i \in I\}, \tag{3}$$

*where* $j_{i,k}$ $(i \in I, k \in \{1, \ldots, N\})$ *is defined in Lemma 23.*

Proof  Let $\lambda = \sum_{i\in I} m_i\Lambda_i = \sum_{i\in I} \lambda_i\epsilon_i \in P^+$. Set $\mathcal{F}$ to the right set of (3).

First, we show $\Psi(\mathbb{H}(\lambda)) \subset \mathcal{F}$. Let $H = H_1 \otimes \cdots \otimes H_N \in \Psi(\mathbb{H}(\lambda))$, where $H_k \in \mathbb{H}(\Lambda_{l_k})$ for $k = 1, 2, \ldots, N$. We know $\lambda_i = m_i + m_{i+1} + \cdots + m_{n-1}$ for $i \in I$. Then by the construction of $\Psi$, $\Lambda_{l_{\lambda_{N+1-i}}} = \Lambda_{N+1-i}$. By Lemma 14, $j_{i,\lambda_{N+1-i}} > 0$ holds. By Lemma 23, $j_{i,\lambda_{N+1-i}} \geq j_{i,\lambda_{N+1-i}+1} \geq \cdots \geq j_{i,N}$ holds. Thus, $H \in F$ holds.

Next, we show $\mathcal{F} \subset \mathbb{H}(\lambda)$. Let $H = H_1\otimes\cdots\otimes H_N \in \bigotimes_{k=1}^{N} \mathbb{H}(\Lambda_{l_k})$, where $H_k = (\Lambda_{l_k}, \mu^{(k)}, 0, (U_{ij}^{(k)})_{i<j})$ for $k = 1, 2, \ldots, N$. Let $\tilde{H} = (\tilde{\lambda}, \tilde{\mu}, 0, (\tilde{U}_{ij})_{i<j})$, where $\tilde{\lambda} = \sum_{k=1}^{N} \Lambda_{l_k}$, $\tilde{\mu} = \sum_{k=1}^{N} \mu^{(k)}$, and $\tilde{U}_{ij} = \sum_{k=1}^{N} U_{ij}^{(k)}$ $(1 \leq i < j \leq n)$. Then we can check $\tilde{H} \in \mathbb{H}(\lambda)$ as follows. For $i \in I$,

$$\begin{aligned} \tilde{\mu}_i &= \sum_{k=1}^{N} \mu_i^{(k)} \\ &= \sum_{k=1}^{N} \left( \sum_{l=1}^{i-1} U_{li}^{(k)} + \left( (\Lambda_{l_k})_i - \sum_{l=i+1}^{n} U_{il}^{(k)} \right) \right) \\ &= \sum_{l=1}^{i-1} \tilde{U}_{li}^{(k)} + \left( \tilde{\lambda}_i^{(k)} - \sum_{l=i+1}^{n} \tilde{U}_{il}^{(k)} \right). \end{aligned}$$

Then $\tilde{H}$ is an integer hive. $\tilde{\lambda} \in P^+$, $\tilde{\mu} \in P$, $\sum_{i\in I} \tilde{\lambda}_i = \sum_{i\in I} \tilde{\mu}_i$, and $\tilde{U}_{ij} \geq 0\,(1 \leq i < j \leq n)$ immediately hold from the definition of $\tilde{H}$ and $H_k \in \mathbb{H}(\Lambda_{l_k})$. For $1 \leq i < j \leq n$,

$$\begin{aligned} \tilde{L}_{ij} &= \sum_{k=1}^{j-1} \tilde{U}_{ik} - \sum_{k=1}^{j} \tilde{U}_{i+1,k} \\ &= \sum_{k=1}^{j-1} \sum_{l=1}^{N} U_{ik}^{(l)} - \sum_{k=1}^{j} \sum_{l=1}^{N} U_{i+1,k}^{(l)} \\ &= \sum_{l=1}^{N} L_{ij}^{(l)} \geq 0. \end{aligned}$$

Also, for $i \in I$,

$$\begin{aligned} \tilde{\mu}_i - \sum_{k=1}^{i-1} \tilde{U}_{ki} &= \sum_{l=1}^{N} \mu_i^{(l)} - \sum_{k=1}^{i-1} \sum_{l=1}^{N} U_{ki}^{(l)} \\ &= \sum_{l=1}^{N} (\mu_i^{(l)} - \sum_{k=1}^{i-1} U_{ki}^{(l)}) \geq 0. \end{aligned}$$

By the choice of $H$, $\tilde{\lambda} = \lambda$. Then $\tilde{H} \in \mathbb{H}(\lambda)$.

We may assume $\Psi(\tilde{H}) = \tilde{H}_1 \otimes \cdots \otimes \tilde{H}_N$, where $\tilde{H}_k = (\Lambda_{l_k}, \tilde{\mu}^{(k)}, 0, (\tilde{U}_{ij}^{(k)})_{i<j})$ for $k = 1, 2, \ldots, N$. We show $\tilde{H}_k = H_k$ for $k = 1, \ldots, N$ by induction on $k$. Set $\tilde{H}^{(N)} = \tilde{H}$ and $\lambda^{(N)} = \lambda$. By Definition 18, we know $\Psi_{\lambda^{(k)}}(\tilde{H}^{(k)}) = \tilde{H}^{(k-1)} \otimes \tilde{H}_k$, where $\tilde{H}^{(k)} = (\lambda^{(k)}, \tilde{\mu}^{(k)}, 0, (V_{ij}^{(k)})_{i<j})$ for $k = 1, 2, \ldots, N$. By Definition 18 and $H \in \mathcal{F}$,

$$\begin{aligned}\tilde{U}_{ij}^{(N)} &= \begin{cases} 1 & \text{if } j = \min\{j \in [n] \mid U_{ij}^{(1)} + \cdots + U_{ij}^{(N)} > 0\}, \\ 0 & \text{otherwise,} \end{cases} \\ &= \begin{cases} 1 & \text{if } j = j_{i,N}, \\ 0 & \text{otherwise,} \end{cases} \\ &= U_{ij}^{(N)}.\end{aligned}$$

By Definition 18, $\tilde{\mu}^{(N)} = \mu^{(N)}$, namely $\tilde{H}_N = H_N$ holds. Assume that $\tilde{H}_s = H_s$ for $s = k+1, k+2, \ldots, N$. By Definition 18, $H \in \mathcal{F}$, and the induction hypothesis,

$$\begin{aligned}\tilde{U}_{ij}^{(k)} &= \begin{cases} 1 & \text{if } j = \min\{j \in [n] \mid U_{ij}^{(1)} + \cdots + U_{ij}^{(k)} > 0\}, \\ 0 & \text{otherwise,} \end{cases} \\ &= \begin{cases} 1 & \text{if } j = j_{i,k}, \\ 0 & \text{otherwise,} \end{cases} \\ &= U_{ij}^{(k)}.\end{aligned}$$

By Definition 18, $\tilde{\mu}^{(k)} = \mu^{(k)}$, namely $\tilde{H}_k = H_k$ holds. Thus, $H \in \Psi(\mathbb{H}(\lambda))$. ∎

**Lemma 26**
*Let $H = (\Lambda_\nu, \mu, 0, (U_{ij})_{i<j}) \in \mathbb{H}(\Lambda_\nu)$. Suppose that there exists $i_0, j_0, i_1, j_1 \in [n]$ such that $U_{i_0,j_0}, U_{i_1,j_1} > 0$. Then $i_1 > i_0$ if and only if $j_1 > j_0$.*

Proof Let $H = (\Lambda_\nu, \mu, 0, (U_{ij})_{i<j}) \in \mathbb{H}(\Lambda_\nu)$. Suppose that there exists $i_0, j_0, i_1, j_1 \in [n]$ such that $U_{i_0,j_0}, U_{i_1,j_1} > 0$.

Assume $i_1 > i_0$ and $i_1 = i_0 + l$ for some $l \in \mathbb{Z}$. By Lemma 14 and $H \in \mathbb{H}(\Lambda_\nu)$,

$$\begin{aligned}\sum_{k=0}^{l-1} L_{i_0+k, j_0+k} &= \sum_{k=1}^{j_0-1} U_{i_0,k} - \sum_{k=1}^{j_0+l-1} U_{i_0+l,k} \\ &= -\sum_{k=1}^{j_0+l-1} U_{i_0+l,k} \geq 0.\end{aligned}$$

Then, we have $U_{i_1k} = 0$ for $k = 1, 2, \ldots, j_0 + l - 1$, especially $U_{i_1k} = 0$ if $k \leq j_0$. Thus, $j_1 > j_0$ holds.

Assume $j_1 > j_0$. Suppose that $i_0 \geq i_1$ and $i_0 = i_1 + l$ for some $l \in \mathbb{Z}$. By Lemma 14 and $H \in \mathbb{H}(\Lambda_\nu)$,

$$\begin{aligned}\sum_{k=0}^{l-1} L_{i_1+k, j_1+k} &= \sum_{k=1}^{j_1-1} U_{i_1k} - \sum_{k=1}^{j_1+l-1} U_{i_1+l,k} \\ &= -\sum_{k=1}^{j_1+l-1} U_{i_1+l,k} \geq 0\end{aligned}$$

Then, we have $U_{i_0k} = 0$ for $k = 1, 2, \ldots, j_1 + l - 1$, especially $U_{i_0k} = 0$ if $k < j_1$, however, this is a contradiction for $j_1 > j_0$. Thus, $i_1 > i_0$ holds. ∎

**Remark 27**
*For $H \in \mathbb{H}(\lambda)$ $(\lambda \in P^+)$, let $\Psi(H) = H_1 \otimes \cdots \otimes H_N$, where $H_k = (\Lambda_{l_k}, \mu^{(k)}, 0, (U^{(k)}_{ij})_{i<j}) \in \mathbb{H}(\Lambda_k)$ for $k = 1, 2, \ldots, N$. For $i \in [n]$ and $k \in \{1, 2, \ldots, N\}$, let $j_{i,k}$ be as in Lemma 23. Then, for each $k = 1, 2, \ldots, N$, we have*

$$j_{1,k} < j_{2,k} < \cdots < j_{l_k,k} \tag{4}$$

*from Lemma 26.*

**Proposition 28**
*Let $\lambda \in P$. $\Psi(\mathbb{H}(\lambda)) \cup \{0\}$ is stable under the action of $e_i$ and $f_i$ for $i \in I$.*

Proof We show that $f_i(\Psi(\mathbb{H}(\lambda)) \cup \{0\}) \subset \Psi(\mathbb{H}(\lambda)) \cup \{0\}$. Let $H = H_1 \otimes \cdots \otimes H_N \in \Psi(\mathbb{H}(\lambda))$, where $H_k = (\Lambda_{l_k}, \mu^{(k)}, 0, (U^{(k)}_{ij})_{i<j})$. Assume $f_iH = H_1 \otimes \cdots \otimes f_iH_{k_0} \otimes \cdots \otimes H_N$. If $f_iH = 0$, the statement is obvious.

Suppose $f_iH \neq 0$. Let $f_iH_{k_0} = (\Lambda_{l_{k_0}}, \tilde{\mu}^{(k_0)}, 0, (\tilde{U}^{(k_0)}_{ij})_{i<j})$. For $i \in I$, if there exists $j \in [n]$ such that $\tilde{U}^{(k_0)}_{ij} > 0$, then set $\tilde{j}_{i,k_0}$ to its $j$, otherwise set $\tilde{j}_{i,k_0}$ to 0. For $H_{k_0}$ and $i$, let $k_0$ in Definition 15 (5) be written as $k_{f_iH}$. Then we know $j_{k_{f_iH},k_0} = i$. By Definition 15, we have $\tilde{j}_{k_{f_iH},k_0} = i + 1$ and $\tilde{j}_{k,k_0} = j_{k,k_0}$ if $k \neq k_{f_iH}$. By Proposition 25, to show that $f_iH \in \Psi(H)$, it suffices to check that $j_{k_{f_iH},k_0-1} \geq \tilde{j}_{k_{f_iH},k_0} = i + 1$. Note that we have $j_{k_{f_iH},k_0-1} \geq j_{k_{f_iH},k_0} = i$ since $H \in \Psi(\mathbb{H}(\lambda))$. It also follows that $\varphi_i(H_{k_0-1}) = 0$ since $\varphi_i(H_{k_0-1}) - \varepsilon_i(H_{k_0}) \leq 0$ holds from Proposition 4.

Suppose $j_{k_{f_iH},k_0-1} = i$. Then, $\mu^{(k_0-1)}_i = \mu^{(k_0-1)}_{i+1} = 1$ follows from Remark 16 and $\varphi_i(H_{k_0-1}) = 0$. By Lemma 26, $j_{k_{f_iH}+1,k_0-1} = i+1$ and $j_{k_{f_iH}+1,k_0} > i$ holds. Since $f_iH^{(k_0)} \neq 0$, we know $\mu^{(k_0)}_{i+1} = 0$ by Remark 16. Then, we have $j_{k_{f_iH}+1,k_0} > i+1$ from (2). Now, we have $j_{k_{f_iH}+1,k_0-1} = i+1 < j_{k_{f_iH}+1,k_0}$, however this is a contradiction for $H \in \Psi(\mathbb{H}(\lambda))$. Thus, $j_{k_{f_iH},k_0-1} \geq i + 1$ holds.

Similarly, $e_i(\Psi(\mathbb{H}(\lambda)) \cup \{0\}) \subset \Psi(\mathbb{H}(\lambda)) \cup \{0\}$ is can be shown. ∎

Now, we define the crystal structure on $\mathbb{H}(\lambda)$ $(\lambda \in P^+)$ using an injection $\Psi$.

**Definition 29 ([14])**
*Let $\lambda = \sum_{i \in I} m_i\Lambda_i \in P^+$. The crystal structure on $\mathbb{H}(\lambda)$ is defined so that $\Psi$ is a morphism of crystals.*

The crystal structure on $\mathbb{H}(\lambda)$ is isomorphic to the crystal basis of an irreducible highest weight module of type $A_{n-1}$ as follows.

**Definition 30 ([14])**
*Let $\lambda \in P^+$. Then define $H_\lambda \in \mathbb{H}(\lambda)$ by $H_\lambda = (\lambda, \lambda, 0, (0)_{i<j})$.*

**Remark 31 ([14])**
*Let $\lambda \in P^+$. Let $H_\lambda = (\lambda, \lambda, 0, (0)_{i<j}) \in \mathbb{H}(\lambda)$. For $i = 1, 2, \ldots, \ell(\lambda)$, we have*

$$U_{ii} = \lambda_i - \sum_{k=1}^{i-1} U_{ki} = \lambda_i > 0.$$

**Lemma 32 ([14])**
*Let $\lambda \in P^+$. Then $H_\lambda$ is the highest weight element of weight $\lambda$ in $\mathbb{H}(\lambda)$.*

**Lemma 33 ([14])**
*Let $\lambda = \sum_{i \in I} m_i \Lambda_i \in P^+$. Then we have*

$$\mathbb{H}(\lambda) = \{f_{i_1} \dots f_{i_k} H_\lambda \mid k \geq 0, i_1, \dots, i_k \in I\}.$$

*Therefore $\mathbb{H}(\lambda)$ is connected.*

For $\lambda \in P^+$, let $B(\lambda)$ be the crystal basis of $V(\lambda)$ with the highest weight vector $b_\lambda$. For an arbitrary $\lambda \in P^+$, to show that $\mathbb{H}(\lambda)$ is isomorphic to $B(\lambda)$, we first show that $\mathbb{H}(\Lambda_\nu)$ is isomorphic to $B(\Lambda_\nu)$ for $\nu \in I$.

**Proposition 34**
*Let $H, H' \in \mathbb{H}(\Lambda_\nu)$. If $\mathrm{wt}(H) = \mathrm{wt}(H')$, then $H = H'$ holds.*

Proof Let $H = (\Lambda_\nu, \mu, 0, (U_{ij})_{i<j}) \in \mathbb{H}(\Lambda_\nu)$. Set $\lambda = \Lambda_\nu$. For $s = 1, 2, \dots, \nu$, there exists a unique $j_s \in [n]$ such that $U_{sj_s} = 1$ by Lemma 14. By Lemma 14 and (2), $\mu_k = 1$ if $k = j_s$ for some $s = 1, 2, \dots, \nu$, otherwise $\mu_k = 0$. By Lemma 26, we have $j_1 < j_2 < \cdots < j_\nu$. Thus, $(s, j_s)$ is uniquely determined by $\lambda$ and $\mu$. Therefore, if $\mathrm{wt}(H) = \mathrm{wt}(H')$, then $H = H'$ holds for $H, H' \in \mathbb{H}(\Lambda_\nu)$. ∎

By the proof of Proposition 34, we have the following.

**Corollary 35**
*Let $H = (\Lambda_\nu, \mu, 0, (U_{ij})_{i<j}) \in \mathbb{H}(\Lambda_\nu)$. For $s = 1, 2, \dots, \nu$, let $j_s \in [n]$ such that $\mu_{j_s} = 1$. Assume $j_1 < j_2 < \cdots < j_\nu$. Then,*

$$U_{ij} = \begin{cases} 1 & \text{if } (i, j) = (s, j_s), \\ 0 & \text{otherwise.} \end{cases}$$

**Lemma 36**
*For $H = (\Lambda_\nu, \mu, 0, (U_{ij})_{i<j}) \in \mathbb{H}(\Lambda_\nu)$, set $\Omega(H) = (\Lambda_\nu, \xi, 0, (V_{ij})_{i<j})$, where $\xi_i = \mu_{n+1-i}$ $(i \in [n])$ and $V_{ij} = U_{\nu+1-i, n+1-j}$ $(1 \leq i < j \leq n)$. Then, $\Omega(H) \in \mathbb{H}(\Lambda_\nu)$.*

Proof Set $\lambda = \Lambda_\nu$. For $s = 1, 2, \dots, \nu$, we can take $j_s \in [n]$ such that $\mu_{j_s} = 1$ since $H \in \mathbb{H}(\Lambda_\nu)$. We may assume $j_1 < j_2 < \cdots < j_\nu$ by retaking $j_s$ if necessary. By Corollary 35,

$$U_{ij} = \begin{cases} 1 & \text{if } (i, j) = (s, j_s) \text{ for some } s \in \{1, 2, \dots, \nu\}, \\ 0 & \text{otherwise.} \end{cases}$$

By the definition of $\Omega$, $\xi_k = 1$ if $k = n + 1 - j_s$, otherwise $\xi_k = 0$. Also, we have

$$\begin{aligned} V_{ij} &= U_{\nu+1-i, n+1-j} \\ &= \begin{cases} 1 & \text{if } (i, j) = (\nu + 1 - s, n + 1 - j_s), \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Since $\xi \in P$ and $\sum_{i=1}^{n} \xi_i = \nu$, we can take $H' \in \mathbb{H}(\Lambda_\nu)$ such that $\mathrm{wt}(H') = \xi$. By Corollary 35, $\Omega(H) = H'$ holds, and hence $\Omega(H) \in \mathbb{H}(\Lambda_\nu)$ holds. ∎

**Definition 37**
*The map $\Omega\colon \mathbb{H}(\Lambda_\nu) \cup \{0\} \to \mathbb{H}(\Lambda_\nu) \cup \{0\}$ is defined by $H$ maps to $\Omega(H)$ for $H \in \mathbb{H}(\Lambda_\nu)$ and $\Omega(0) = 0$.*

**Proposition 38**
*The map* $\Omega\colon \mathbb{H}(\Lambda_\nu) \cup \{0\} \to \mathbb{H}(\Lambda_\nu) \cup \{0\}$ *is an involution.*

Proof  Let $H \in \mathbb{H}(\Lambda_\nu)$. By Definition 37, we have $\Omega(\Omega(H)) = H$. Also, we have $\Omega(0) = 0$. Then, $\Omega$ is a surjection. Let $H, K \in \mathbb{H}(\Lambda_\nu) \cup \{0\}$. Assume $\Omega(H) = \Omega(K)$. By Definition 37, we have $H = \Omega(\Omega(H)) = \Omega(\Omega(K)) = K$. Then $\Omega$ is an injection. Thus, $\Omega$ is a bijection, especially $\Omega$ is an involution. ∎

**Proposition 39**
$\Omega\colon \mathbb{H}(\Lambda_\nu) \to \mathbb{H}(\Lambda_\nu)$ *has the following properties. For* $H \in \mathbb{H}(\Lambda_\nu)$ *and* $i \in I$*,*

1. $\mathrm{wt}(\Omega(H)) = w_0 \mathrm{wt}(H)$,
2. $\varphi_i(\Omega(H)) = \varepsilon_{n-i}(H)$,
3. $\varepsilon_i(\Omega(H)) = \varphi_{n-i}(H)$,
4. $f_i(\Omega(H)) = \Omega(e_{n-i}(H))$,
5. $e_i(\Omega(H)) = \Omega(f_{n-i}(H))$,

*where* $w_0$ *denotes the longest element in the Weyl group of type* $A_{n-1}$.

Proof  Let $H = (\Lambda_\nu, \mu, 0, (U_{ij})_{i<j}) \in \mathbb{H}(\Lambda_\nu)$. Let $w_0$ be the longest element in the Weyl group of type $A_{n-1}$. By Definition 37, we have

$$\begin{aligned}\mathrm{wt}(\Omega(H)) &= \sum_{k=1}^{n} \mu_{n+1-k}\epsilon_k = \sum_{k=1}^{n} \mu_k \epsilon_{n+1-k} \\ &= \sum_{k=1}^{n} \mu_k w_0(\epsilon_k) = w_0 \mathrm{wt}(H),\end{aligned}$$

hence (1) holds.

By Definition 37, we have

$$\begin{aligned}\varphi_i(\Omega(H)) &= \max\{\mu_{n+1-i} - \mu_{n-i}, 0\} \\ &= \varepsilon_{n-1}(H).\end{aligned}$$

Then (2) holds. Also, we have

$$\begin{aligned}\varepsilon_i(\Omega(H)) &= \max\{\mu_{n-i} - \mu_{n+1-i}, 0\} \\ &= \varphi_{n-1}(H).\end{aligned}$$

Then (3) holds.

From (2), (4) is obvious if $f_i\Omega(H) = 0$. Suppose $f_i\Omega(H) \neq 0$. Set $\xi = \mathrm{wt}(f_i\Omega(H))$ and $o = \mathrm{wt}(\Omega(e_{n-i}(H)))$. By Definitions 15 and 37, for $k = 1, 2, \ldots, n$,

$$\begin{aligned}\xi_k &= \begin{cases} \mu_{n+1-k} - 1 & \text{if } k = i, \\ \mu_{n+1-k} + 1 & \text{if } k = i+1, \\ \mu_{n+1-k} & \text{otherwise} \end{cases} \\ &= o_k.\end{aligned}$$

By Proposition 34, (4) holds.

From (3), (5) is obvious if $e_i\Omega(H) = 0$. Suppose $e_i\Omega(H) \neq 0$. Set $\xi = \mathrm{wt}(e_i\Omega(H))$ and $o = \mathrm{wt}(\Omega(f_{n-i}(H)))$. By Definitions 15 and 37, for $k = 1, 2, \ldots, n$,

$$\begin{aligned}\xi_k &= \begin{cases}\mu_{n+1-k} + 1 & \text{if } k = i,\\ \mu_{n+1-k} - 1 & \text{if } k = i+1,\\ \mu_{n+1-k} & \text{otherwise}\end{cases}\\ &= o_k.\end{aligned}$$

By Proposition 34, (5) holds. ∎

**Proposition 40**
*Let $k \in I$. There is an isomorphism from $\mathbb{H}(\Lambda_k)$ to $B(\Lambda_k)$.*

Proof Let $k \in I$. From [15][1, Theorem 4.13], it suffices to show that

1. If $e_i(H) = 0$, then $\varepsilon_i(H) = 0$ for $H \in \mathbb{H}(\Lambda_k)$, $i \in I$,
2. If $f_i(H) = 0$, then $\varphi_i(H) = 0$ for $H \in \mathbb{H}(\Lambda_k)$, $i \in I$,
3. When $i, j \in I$ and $i \neq j$, if $H, K \in \mathbb{H}(\Lambda_k)$ and $K = e_iH$, then $\varepsilon_j(K)$ equals $\varepsilon_j(H)$ or $\varepsilon_j(H) + 1$. The second case where $\varepsilon_j(K) = \varepsilon_j(H) + 1$ is possible only if $\alpha_i$ and $\alpha_j$ are not orthogonal roots,
4. When $i, j \in I$ and $i \neq j$, if $H, K \in \mathbb{H}(\Lambda_k)$ and $K = f_iH$, then $\varphi_j(K)$ equals $\varphi_j(H)$ or $\varphi_j(H) + 1$. The second case where $\varphi_j(K) = \varphi_j(H) + 1$ is possible only if $\alpha_i$ and $\alpha_j$ are not orthogonal roots,
5. Assume that $i, j \in I$ and $i \neq j$. If $H \in \mathbb{H}(\Lambda_k)$ with $\varepsilon_i(H) > 0$ and $\varepsilon_j(e_iH) = \varepsilon_j(H) > 0$, then $e_ie_jH = e_je_iH$ and $\varphi_i(e_jH) = \varphi_i(H)$,
6. Assume that $i, j \in I$ and $i \neq j$. If $H \in \mathbb{H}(\Lambda_k)$ with $\varphi_i(H) > 0$ and $\varphi_j(f_iH) = \varphi_j(H) > 0$, then $f_if_jH = f_jf_iH$ and $\varepsilon_i(f_jH) = \varepsilon_i(H)$,
7. Assume that $i, j \in I$ and $i \neq j$. If $H \in \mathbb{H}(\Lambda_k)$ with $\varepsilon_j(e_iH) = \varepsilon_j(H) + 1 > 1$ and $\varepsilon_i(e_jH) = \varepsilon_i(H) + 1 > 1$, then $e_ie_j^2e_iH = e_je_i^2e_jH \neq 0$, $\varphi_i(e_jH) = \varphi_i(e_j^2e_iH)$ and $\varphi_j(e_iH) = \varphi_j(e_i^2e_jH)$,
8. Assume that $i, j \in I$ and $i \neq j$. If $H \in \mathbb{H}(\Lambda_k)$ with $\varphi_j(f_iH) = \varphi_j(H) + 1 > 1$ and $\varphi_i(f_jH) = \varphi_i(H) + 1 > 1$, then $f_if_j^2f_iH = f_jf_i^2f_jH \neq 0$, $\varepsilon_i(f_jH) = \varepsilon_i(f_j^2f_iH)$ and $\varepsilon_j(f_iH) = \varepsilon_j(f_i^2f_jH)$.

by Remark 16, Lemmas 33 and 32. By Remark 16, (1) and (2) hold. Also, again by Remark 16, we know that there is no $i \in I$ such that $\varepsilon_i(H) > 1$ (resp. $\varphi_i(H) > 1$), so (7) (resp. (8)) is true.

Let $i, j \in I$ with $i \neq j$. Let $H, K \in \mathbb{H}(\Lambda_k)$. Assume $K = e_iH$. By Definition 15, $\varepsilon_j(K) = \varepsilon_j(H)$ is obvious if $j \neq i-1, i+1$. Let $H = (\Lambda_k, \mu, 0, (U_{ij})_{i<j})$ and $K = (\Lambda_k, \xi, 0, (V_{ij})_{i<j})$. We know $\varepsilon_i(H) = 1$ from $K \neq 0$ and Remark 16, especially $\mu_{i+1} = 1$ and $\mu_i = 0$. By Definition 15, if $\mu_{i-1} = 0$, then $\varepsilon_{i-1}(K) = \varepsilon_{i-1}(H) + 1$, otherwise $\varepsilon_{i-1}(K) = \varepsilon_{i-1}(H)$. Also, if $\mu_{i+2} = 1$, then $\varepsilon_{i+1}(K) = \varepsilon_{i+1}(H) + 1$, otherwise $\varepsilon_{i+1}(K) = \varepsilon_{i+1}(H)$. Then (3) holds.

Let $i, j \in I$ with $i \neq j$. Let $H \in \mathbb{H}(\Lambda_k)$. Assume that $\varepsilon_i(H) > 0$ and $\varepsilon_j(e_iH) = \varepsilon_j(H) > 0$. By Definition 15, $\mathrm{wt}(e_ie_jH) = \mathrm{wt}(e_je_iH)$ holds. Then, $e_ie_jH = e_je_iH$ holds by Proposition 34. By assumption and (3), we can assume $j \neq i-1, i+1$. Then, we have $\varphi_i(e_jH) = \varphi_i(H)$ by Definition 15. Thus, (5) is satisfied.

By Propositions 38, 39, and (5), (6) immediately holds. ∎

Then we have the following from Definition 29 and Proposition 40.

**Theorem 41 ([14])**
*Let $\lambda \in P^+$. Then, we have a crystal isomorphism $\Phi\colon \mathbb{H}(\lambda) \to B(\lambda)$ such that $\Phi(H_\lambda) = b_\lambda$.*

The crystal structure on $\mathbb{H}(\lambda)$ can also be given by considering a combinatorial description.

**Theorem 42 ([14])**
*Let $\lambda = \sum_{i\in I} m_i\Lambda_i$. For $H \in \mathbb{H}(\lambda)$, the maps wt, $f_j$, $e_j$, $\varphi_j$, $\varepsilon_j$ ($j \in I$) are computed as follows. Fix $j \in I$.*

1. $\mathrm{wt}(H) = \sum_{i\in I}(\mu_i - \mu_{i+1})\Lambda_i$.
2. *For $k \in \{1, 2, \ldots, j\}$, set $\varphi_j^{(k)}(H) = \max\{\varphi_j^{(k-1)}(H) + U_{k,j} - U_{k+1,j+1}, 0\}$. Note that we regard $\varphi_j^{(0)}$ as 0. Then, we have $\varphi_j(H) = \varphi_j^{(j)}(H)$.*
3. *For $k \in \{1, 2, \ldots, j+1\}$, set $\varepsilon_j^{(k)}(H) = \max\{\varepsilon_j^{(k-1)}(H) + U_{j+2-k,j+1} - U_{j+1-k,j}, 0\}$. Note that we regard $\varepsilon_j^{(0)}$ as 0. Then, we have $\varepsilon_j(H) = \varepsilon_j^{(j+1)}(H)$.*
4. *If $\varphi_j(H) = 0$ then $f_jH = 0$. If $\varphi_j(H) \neq 0$, let*
$$k_{f_jH} = \min\{k \in [n] \mid \forall l \geq k,\ \varphi_j^{(l)}(H) > 0\}.$$
*Then, we have $f_jH = (\lambda, \mu', 0, (U'_{kl})_{k<l})$ where*
$$\mu' = \sum_{k\neq j,j+1} \mu_k\epsilon_k + (\mu_j - 1)\epsilon_j + (\mu_{j+1} + 1)\epsilon_{j+1},$$
$$U'_{kl} = \begin{cases} U_{kl} - 1 & \text{if } k = k_{f_jH},\ l = j, \\ U_{kl} + 1 & \text{if } k = k_{f_jH},\ l = j+1, \\ U_{kl} & \text{else.} \end{cases}$$
5. *If $\varepsilon_j(H) = 0$ then $e_jH = 0$. If $\varepsilon_j(H) \neq 0$, let*
$$k_{e_jH} = \min\{k \in [n] \mid \forall l \geq k,\ \varepsilon_j^{(l)}(H) > 0\}.$$
*Then, we have $e_jH = (\lambda, \mu', 0, (U'_{kl})_{k<l})$ where*
$$\mu' = \sum_{k\neq j,j+1} \mu_k\epsilon_k + (\mu_j + 1)\epsilon_j + (\mu_{j+1} - 1)\epsilon_{j+1},$$
$$U'_{kl} = \begin{cases} U_{kl} + 1 & \text{if } k = j+2-k_{e_jH},\ l = j, \\ U_{kl} - 1 & \text{if } k = j+2-k_{e_jH},\ l = j+1, \\ U_{kl} & \text{else.} \end{cases}$$

# 3 Algorithms for the Crystal Structure on K-hives

In this section, we give a set of algorithms to compute the components of the crystal structure on $\mathbb{H}(\lambda)$ ($\lambda \in P^+$) using two approaches. One approach is based on Definition 29, which implies that the crystal structure on $\mathbb{H}(\lambda)$ is regarded as a subset of a tensor product of the form $\mathbb{H}(\Lambda_k)$ with $k \in I$. The other approach is based on Theorem 42, which is a more combinatorial description.

To consider algorithms, we regard $H = (\lambda, \mu, 0, (U_{ij})_{i<j}) \in \mathbb{H}(\lambda)$ as a hash table with keys $\lambda$, $\mu$, $\gamma$, and $(U_{ij})_{i<j}$, where the value of $\lambda$ is an array $[\lambda_1, \lambda_2, \ldots, \lambda_n]$, the value of $\mu$ is an

array $[\mu_1, \mu_2, \ldots, \mu_n]$, the value of $\gamma$ is an array $[0, 0, \ldots, 0]$, and the value of $(U_{ij})_{i<j}$ is a two-dimensional array $[[U_{12}, U_{13}, \ldots], [U_{23}, \ldots], \ldots, [U_{n-1,n}]]$.

To give algorithms for the crystal structure on $\mathbb{H}(\lambda)$ based on Definition 29, we first consider algorithms for the crystal structure on $\mathbb{H}(\Lambda_k)$ $(k \in I)$. The maps $f_i$ (resp. $e_i$) $(I \in I)$ for $\mathbb{H}(\Lambda_k)$ $(k \in I)$ are computed by Algorithm 1 (resp. Algorithm 2). Note that the maps wt, $\varphi_i, \varepsilon_i$ $(i \in I)$ are simply computed by Definition 15 as $\sum_{k\in I}(\mu_k - \mu_{k+1})\Lambda_k$, $\max(\mu_i - \mu_{i+1}, 0)$, $\max(\mu_{i+1} - \mu_i, 0)$, respectively for $H = (\Lambda_k, \mu, 0, (U_{ij})_{i<j}) \in \mathbb{H}(\Lambda_k)$.

**Algorithm 1** Algorithm for $f_i$ on $\mathbb{H}(\Lambda_k)$

**Input:** $H = (\Lambda_k, \mu, 0, (U_{ij})_{i<j}) \in \mathbb{H}(\Lambda_k)$, $i \in I$
**Output:** $f_i H$

**if** $\max(\mu_i - \mu_{i+1}, 0) = 0$ **then**
    **return** 0
**end if**
Take $k_0$ from $\{k \in [i] \mid U_{k,i} > 0\}$
$\mu_i := \mu_i - 1$
$\mu_{i+1} := \mu_{i+1} + 1$
$U_{k_0,i} := U_{k_0,i} - 1$
$U_{k_0,i+1} := U_{k_0,i+1} + 1$
**return** $(\Lambda_k, \mu, 0, (U_{ij})_{i<j})$

**Algorithm 2** Algorithm for $e_i$ on $\mathbb{H}(\Lambda_k)$

**Input:** $H = (\Lambda, \mu, 0, (U_{ij})_{i<j}) \in \mathbb{H}(\Lambda_k)$, $i \in I$
**Output:** $e_i H$

**if** $\max(\mu_{i+1} - \mu_i, 0) = 0$ **then**
    **return** 0
**end if**
Take $k_0$ from $\{k \in [i+1] \mid U_{k,i+1} > 0\}$
$\mu_i := \mu_i + 1$
$\mu_{i+1} := \mu_{i+1} - 1$
$U_{k_0,i} := U_{k_0,i} + 1$
$U_{k_0,i+1} := U_{k_0,i+1} - 1$
**return** $(\lambda, \mu, 0, (U_{ij})_{i<j})$

Let us give an example of executing Algorithm 1.

**Example 43**
*The action of $f_i$ on the $U_q(\mathfrak{sl}_4)$-crystal $\mathbb{H}(\Lambda_3)$ is computed as follows by Algorithm 1. Let $H = (\Lambda_3, \Lambda_3, 0, (0)_{k<l}) \in \mathbb{H}(\Lambda_3)$. Set $\mu = \Lambda_3$. Note that $\Lambda_3$ is represented by the partition $(1, 1, 1, 0)$. For $i = 1$, we have $f_1 H = 0$ since $\max(\mu_1 - \mu_2, 0) = 0$. Also, for $i = 2$, we have $f_2 H = 0$ since $\max(\mu_2 - \mu_3, 0) = 0$. Let $i = 3$. In this case, $\max(\mu_3 - \mu_4, 0) = 1$. Then we can proceed to the next step. Since $\{k \in [3] \mid U_{k,3} > 0\} = \{3\}$, $k_0$ is uniquely determined to 3. Then set $\xi = \mu$, then set $\xi_3 = \mu_3 - 1 = 0$ and $\xi_4 = \mu_4 + 1 = 1$. Also, set $V_{ij} = U_{ij}$, and set $V_{3,3} = U_{3,3} - 1 = 0$ and $V_{3,4} = U_{3,4} + 1 = 1$. Then we have $f_i H = (\Lambda_3, \xi, 0, (V_{ij})_{i<j})$. See Fig. 3.*

Algorithms 1 and 2 generate results that correspond to Definition 15 as follows.

**Proposition 44**
*For $k \in I$, let $H \in \mathbb{H}(\Lambda_k)$. Let $i \in I$.*

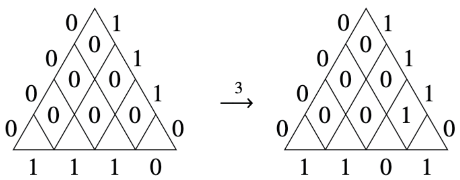



Fig. 3: Action of $f_3$ on the $U_q(\mathfrak{sl}_4)$-crystal $\mathbb{H}(\Lambda_3)$

1. *Let $K$ be the result of Algorithm 1 with inputs $H$ and $i$. Then, $K = f_i H$,*
2. *Let $K$ be the result of Algorithm 2 with inputs $H$ and $i$. Then, $K = e_i H$.*

Proof For $k \in I$, let $H \in \mathbb{H}(\Lambda_k)$. Let $i \in I$. (1) Let $K$ be the result of Algorithm 1 with inputs $H$ and $i$. By Lemma 1, $k_0$ in Algorithm 1 is uniquely determined. Then we have $K = f_i H$ from Definition 15. Similarly, (2) can be shown. ∎

For $\lambda \in P^+$, the map $\Psi_\lambda$ is computed by Algorithm 3.

The following is an example of executing Algorithm 3.

**Example 45**

*Let $n = 4$, $\lambda = (3, 2, 1, 0)$, and $\mu = (2, 3, 1, 0)$. Let $H = (\lambda, \mu, 0, (U_{ij})_{i<j}) \in \mathbb{H}(\lambda)$, where $U_{12} = 1$ and $U_{ij} = 0$ if $(i, j) \neq (1, 2)$ and $i < j$. Then $\Psi_\lambda(H)$ is computed by Algorithm 3 as follows. Set $\nu = \ell(\lambda) = 3$. Let $\lambda^{(2)} = (\lambda^{(2)}_1, \lambda^{(2)}_2, \dots, \lambda^{(2)}_n)$, where $\lambda^{(2)}_k = 1$ if $k \in [\nu]$ else $\lambda^{(2)}_k = 0$. Set $U^{(2)}_{ij} = U_{ij}$ for $1 \leq i < j \leq 4$. Since $\min\{l \in [4] \mid U_{1l} > 0\} = 1$, set $U^{(2)}_{11} = 1$ and $U^{(2)}_{12} = U^{(2)}_{13} = U^{(2)}_{14} = 0$. Since $\min\{l \in [4] \mid U_{2l} > 0\} = 2$, set $U^{(2)}_{22} = 1$ and $U^{(2)}_{23} = U^{(2)}_{24} = 0$. Since $\min\{l \in [4] \mid U_{3l} > 0\} = 3$, set $U^{(2)}_{33} = 1$ and $U^{(2)}_{34} = 0$. Set*

$$\mu^{(2)}_1 = U^{(2)}_{11} = 1, \quad \mu^{(2)}_2 = U^{(2)}_{12} + U^{(2)}_{22} = 1,$$
$$\mu^{(2)}_2 = U^{(2)}_{13} + U^{(2)}_{23} + U^{(2)}_{33} = 1, \quad \mu^{(2)}_4 = U^{(2)}_{14} + U^{(2)}_{24} + U^{(2)}_{34} + U^{(2)}_{44} = 0.$$

*Set*

$$\lambda^{(1)}_1 = \lambda_1 - \lambda^{(2)}_1 = 2, \quad \lambda^{(1)}_2 = \lambda_2 - \lambda^{(2)}_2 = 1,$$
$$\lambda^{(1)}_3 = \lambda_3 - \lambda^{(2)}_3 = 0, \quad \lambda^{(1)}_4 = \lambda_4 - \lambda^{(2)}_4 = 0.$$

*Set $U^{(1)}_{ij} = U_{ij} - U^{(2)}_{ij}$ for $1 \leq i \leq j \leq 4$. Set*

$$\mu^{(1)}_1 = U^{(1)}_{11} = 1, \quad \mu^{(1)}_2 = U^{(1)}_{12} + U^{(1)}_{22} = 2,$$
$$\mu^{(1)}_2 = U^{(1)}_{13} + U^{(1)}_{23} + U^{(1)}_{33} = 0, \quad \mu^{(1)}_4 = U^{(1)}_{14} + U^{(1)}_{24} + U^{(1)}_{34} + U^{(1)}_{44} = 0.$$

*Then $\Psi_\lambda = (\lambda^{(1)}, \mu^{(1)}, 0, (U^{(1)}_{ij})) \otimes (\lambda^{(2)}, \mu^{(2)}, 0, (U^{(2)}_{ij}))$. See Fig. 4*

Algorithm 3 generates a result corresponding to an image of $\Psi_\lambda$.

**Algorithm 3** Algorithm for $\Psi_\lambda$

**Input:** $H = (\lambda, \mu, 0, (U_{ij})_{i<j}) \in \mathbb{H}(\lambda)$
**Output:** $\Psi_\lambda(H)$

**for** $k = 1, 2, \ldots, n$ **do** ▷ Compute $\lambda^{(2)}$
  **if** $k \in [1, \ell(\lambda)]_\mathbb{Z}$ **then**
    $\lambda^{(2)}_k = 1$
  **else**
    $\lambda^{(2)}_k = 0$
  **end if**
**end for**
$\lambda^{(2)} := (\lambda^{(2)}_1, \lambda^{(2)}_2, \ldots, \lambda^{(2)}_n)$
$(U^{(2)}_{ij})_{i<j} := (U_{ij})_{i<j}$ ▷ Compute $(U^{(2)}_{ij})_{i<j}$
**for** $i = 1, 2, \ldots, n-1$ **do**
  **for** $j = i+1, i+2, \ldots, n$ **do**
    **if** $j = \min\{l \in [n] \mid U_{il} > 0\}$ **then**
      $U^{(2)}_{ij} := 1$
    **else**
      $U^{(2)}_{ij} := 0$
    **end if**
  **end for**
**end for**
**for** $k = 1, 2, \ldots, n$ **do** ▷ Compute $\mu^{(2)}$
  $\mu^{(2)}_k := \sum_{l=1}^{i} U^{(2)}_{li}$
**end for**
$\mu^{(2)} := (\mu^{(2)}_1, \mu^{(2)}_2, \ldots, \mu^{(2)}_n)$
**for** $k = 1, 2, \ldots, n$ **do** ▷ Compute $\lambda^{(1)}$
  $\lambda^{(1)}_k := \lambda_k - \lambda^{(2)}_k$
**end for**
$\lambda^{(1)} := (\lambda^{(1)}_1, \lambda^{(1)}_2, \ldots, \lambda^{(1)}_n)$
$(U^{(1)}_{ij})_{i<j} := (U_{ij})_{i<j}$ ▷ Compute $(U^{(1)}_{ij})_{i<j}$
**for** $i = 1, 2, \ldots, n-1$ **do**
  **for** $j = i+1, i+2, \ldots, n$ **do**
    $U^{(1)}_{ij} := U_{ij} - U^{(2)}_{ij}$
  **end for**
**end for**
**for** $i = 1, 2, \ldots, n$ **do** ▷ Compute $\mu^{(1)}$
  $\mu^{(1)}_i = \sum_{l=1}^{i} U^{(1)}_{li}$
**end for**
**return** $(\lambda^{(1)}, \mu^{(1)}, 0, (U^{(1)}_{ij})_{i<j}) \otimes (\lambda^{(2)}, \mu^{(2)}, 0, (U^{(2}_{ij})_{i<j})$

**Proposition 46**
*For $\lambda \in P^+$, let $H \in \mathbb{H}(\lambda)$. Let $K$ be the result of Algorithm 3 with input $H$. Then, $K = \Psi_\lambda(H)$.*

Proof The statement immediately follows from Definition 18. ∎

The map $\Psi$ is defined to apply $\Psi_\lambda$ ($\lambda \in P^+$) repeatedly, and note that the algorithm for $\Psi_\lambda$ is given by Algorithm 3. Then, the map $\Psi$ is computed using Algorithm 4.

The following is an example of executing Algorithm 4.

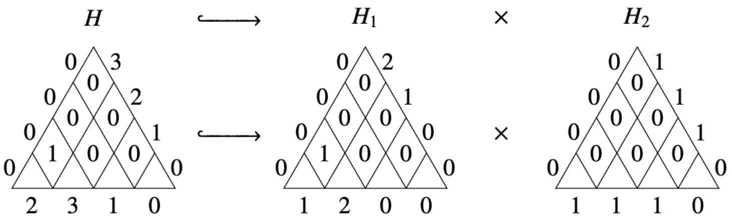



Fig. 4: Action of $\Psi_\lambda$ on $\mathbb{H}(\lambda)$

**Algorithm 4** Algorithm for $\Psi$

**Input:** $H = (\lambda, \mu, 0, (U_{ij})_{i<j}) \in \mathbb{H}(\lambda)$
**Output:** $\Psi(H)$

$H_1 \otimes H_2 := \Psi_\lambda(H)$
$N = 2$
**while** $H_1 \notin \mathbb{H}(\Lambda_k)$ for any $k \in I$ **do**
  $K_1 \otimes K_2 := \Psi(H_1)$
  $H := K_1 \otimes K_2 \otimes H_2 \otimes \cdots \otimes H_N$
  $N = N + 1$
  Rename $H$ as $H = H_1 \otimes H_2 \otimes \cdots \otimes H_N$
**end while**
**return** $\bigotimes_{k \in N} H_k$

**Example 47**
*Let $n = 4$, $\lambda = (3, 2, 1, 0)$ and $\mu = (2, 3, 1, 0)$. Let $H = (\lambda, \mu, 0, (U_{ij})_{i<j}) \in \mathbb{H}(\lambda)$, where $U_{12} = 0$ and $U_{ij} = 0$ if $(i, j) \neq (1, 2)$ and $i < j$. By Algorithm 3,*

$$\begin{aligned}\Psi_\lambda(H) &= ((2,1,0,0),(1,2,0,0),(0^4),(U_{ij}^{(1)})) \otimes ((1,1,1,0),(1,1,1,0),(0^4),(U_{ij}^{(2)})) \\ &:= H_1 \otimes H_2,\end{aligned}$$

*where*

$$U_{ij}^{(1)} = \begin{cases} 1 & \text{if } (i,j) = (1,2), \\ 0 & \text{otherwise,} \end{cases}$$
$$U_{ij}^{(2)} = 0 \quad (1 \leq i < j \leq 4).$$

*Since $H_1 \in \mathbb{H}((2, 1, 0, 0))$, we proceed with the algorithm.*

$$\begin{aligned}\Psi_\lambda(H_1) &= ((1,0,0,0),(0,1,0,0),(0^4),(V_{ij}^1)) \otimes ((1,1,0,0),(1,1,0,0),(0^4),(V_{ij}^1)) \\ &:= K_1 \otimes K_2,\end{aligned}$$

*where*

$$V_{ij}^{(1)} = \begin{cases} 1 & \text{if } (i,j) = (1,2), \\ 0 & \text{otherwise,} \end{cases}$$
$$V_{ij}^{(2)} = 0 \quad (1 \leq i < j \leq 4).$$

*Then rename $K_1 \otimes K_2 \otimes H_2$ as $H_1 \otimes H_2 \otimes H_3$. Then, we have*

$$\Psi(H) = H_1 \otimes H_2 \otimes H_3$$

*See Fig. 5.*

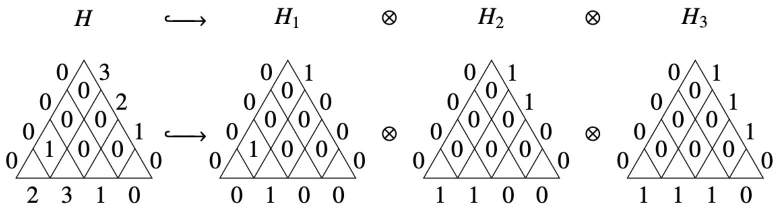



Fig. 5: Action of $\Psi$ on $\mathbb{H}(\lambda)$

The result of Algorithm 4 corresponds to the image of $\Psi$.

**Proposition 48**
*For $\lambda \in P^+$, let $H \in \mathbb{H}(\lambda)$. Let $K$ be the result of Algorithm 4 for input $H$. Then, $K = \Psi(H)$.*

Proof By Proposition 22, it is clear that Algorithm 4 yields the image of $\Psi$ if the while statement stops. For $\lambda \in P^+$, let $H \in \mathbb{H}(\lambda)$. Suppose $H_1 \otimes H_2 \otimes \cdots \otimes H_{k+2}$ is obtained at the $k$-th step of the while statement in Algorithm 4, and $H_1 \notin \mathbb{H}(\Lambda_i)$ for all $i \in I$. Assume $H_1 \in \mathbb{H}(\lambda^{(1)})$ for $\lambda^{(1)} \in P^+$, where $\lambda^{(1)} \neq \Lambda_i$ for all $i \in I$. This means that there exists $m \in [n]$ such that $\lambda^{(1)}_m > 1$, especially $\lambda^{(1)}_1 > 1$. Set $\lambda' = \lambda^{(1)}$ and $m_0 = \lambda^{(1)}_1$. Then at $k + m_0 - 1$ step in the while statement, we have

$$H_1 \otimes H_2 \otimes \cdots \otimes H_{k+m_0+1}.$$

Assume $H_1 \in \mathbb{H}(\lambda^{(1)})$. Note that, since the indices are renamed, we retake $H_1$ and $\lambda^{(1)}$. By Algorithm 4, we have $\lambda^{(1)}_m = \max(\lambda^{(k)}_m - (m_0 - 1), 0)$ for $m \in [n]$. Since $\lambda' \in P^+$ and $m_0 = \lambda'_1$, $\lambda^{(1)}_m \in \{0, 1\}$. Hence $H_1 \in \mathbb{H}(\Lambda_\nu)$ for $\nu \in I$. Thus, the while statement stops. ∎

To compute $f_i, e_i$ $(i \in I)$ on $\mathbb{H}(\lambda)$, we need the algorithm of $\Psi^{-1}$ for the image of $\Psi$. Algorithm 5 computes $\Psi^{-1}$ for $H \in \Psi(\mathbb{H}(\lambda))$.

**Proposition 49**
*For $\lambda \in P^+$, let $H \in \mathbb{H}(\lambda)$. Let $\Psi(H) = H_1 \otimes H_2 \otimes \cdots \otimes H_N$. Let $K$ be the result of Algorithm 5 with input $H_1 \otimes H_2 \otimes \cdots \otimes H_N$. Then, $K = H$.*

Proof For $\lambda \in P^+$, let $H \in \mathbb{H}(\lambda)$. Let $\Psi(H) = H_1 \otimes H_2 \otimes \cdots \otimes H_N$. Let $K$ be the result of Algorithm 5 with input $H_1 \otimes H_2 \otimes \cdots \otimes H_N$. Assume that $H = (\lambda, \mu, 0, (U_{ij})_{i<j})$ and $H_k = (\lambda^{(k)}, \mu^{(k)}, 0, (U^{(k)}_{ij})_{i<j})$ for $k = 1, 2, \ldots, N$. Let $\Psi_\lambda(H) = K_1 \otimes K_2$. Assume $K_m = (\nu^{(m)}, \xi^{(m)}, 0, (V^{(k)}_{ij})_{i<j})$. By Definition 18, we have $\lambda_k = \nu^{(1)}_k + \nu^{(2)}_k$, $\mu_k = \xi^{(1)}_k + \xi^{(2)}_k$ for $k =$

**Algorithm 5** Algorithm for $\Psi^{-1}$

**Input:** $H = H_1 \otimes H_2 \otimes \cdots \otimes H_N \in \bigotimes_k \mathbb{H}(\Lambda_k)$, $H_k = (\lambda^{(k)}, \mu^{(k)}, 0, (U_{ij}^{(k)})_{i<j}) \in \mathbb{H}(\lambda^{(k)})$.
**Output:** $\Psi^{-1}(H) \in \mathbb{H}(\lambda)$

**for** $i = 1, 2, \ldots, n$ **do**
  $\lambda_i := \sum_{k=1}^{N} \lambda_i^{(k)}$
**end for**
$\lambda := (\lambda_1, \lambda_2, \ldots, \lambda_n)$
**for** $i = 1, 2, \ldots, n$ **do**
  $\mu_i := \sum_{k=1}^{N} \mu_i^{(k)}$
**end for**
$\mu := (\mu_1, \mu_2, \ldots, \mu_n)$
**for** $i = 1, 2, \ldots, n-1$ **do**
  **for** $j = i+1, i+2, \ldots, n$ **do**
    $U_{ij} := \sum_{k=1}^{N} U_{ij}^{(k)}$
  **end for**
**end for**
**return** $(\lambda, \mu, 0, (U_{ij})_{i<j})$

$1, 2, \ldots, N$ and $U_{ij} = U_{ij}^{(1)} + U_{ij}^{(2)}$ for $1 \leq i < j \leq n$. Since the construction of $\Psi$, we obtain

$$\lambda_k = \lambda_k^{(1)} + \cdots + \lambda_k^{(N)} \quad (k = 1, 2, \ldots, N),$$
$$\mu_k = \mu_k^{(1)} + \cdots + \mu_k^{(N)} \quad (k = 1, 2, \ldots, N),$$
$$U_{ij} = U_{ij}^{(1)} + \cdots + U_{ij}^{(N)} \quad (1 \leq i < j \leq n).$$

Thus, we have $K = H$. ∎

By Definition 29, the crystal structure on $\mathbb{H}(\lambda)$ is defined by considering $\mathbb{H}(\lambda)$ as a subset of a tensor product of the form $\mathbb{H}(\Lambda_k)$ with $k \in I$. In detail, embedding $H \in \mathbb{H}(\lambda)$ into $\bigotimes_k \mathbb{H}(\Lambda_k)$ by $\Psi$, then compute the maps wt, $\varphi_i, \varepsilon_i, f_i, e_i$ $(i \in I)$ by Definition 3, then pulling it back into $\mathbb{H}(\lambda)$. Then, the maps wt, $\varphi_i, \varepsilon_i, f_i, e_i$ $(i \in I)$ are computed by the following algorithms. For $\lambda \in P^+$, let $H \in \mathbb{H}(\lambda)$. Let $\Psi(H) = H_1 \otimes H_2 \otimes \cdots \otimes H_N$, which is computed by Algorithm 4. Then wt$(H)$ is computed by wt$(H) = \sum_{k=1}^{N}$ wt$(H_k)$, where wt$(H_k)$ is computed by algorithm of wt for $\mathbb{H}(\Lambda_{k'})$ for some $k' \in I$. Then $\varphi_i(H)$ is computed by $\varphi_i(H) = \varphi_i(H_1 \otimes H_2 \otimes \cdots \otimes H_N)$, where $\varphi_i(H_1 \otimes H_2 \otimes \cdots \otimes H_N)$ is computed by Definition 3 and $\varphi_i$ for $\mathbb{H}(\Lambda_k)$ $(k \in I)$. Similarly, $\varepsilon_i(H)$ can be computed. Also, $f_i(H)$ is computed by $\Psi^{-1}(f_i(H_1 \otimes H_2 \otimes \cdots \otimes H_N))$, where $f_i(H_1 \otimes H_2 \otimes \cdots \otimes H_N))$ is computed by Definition 3 and Algorithm 1. Similarly, $e_i(H)$ can be computed.

**Proposition 50**
*Let $\lambda \in P^+$. Let* wt, $\varphi_i, \varepsilon_i, f_i, e_i$ $(i \in I)$ *be computed using the above algorithms for* $\mathbb{H}(\lambda)$*. Then, the crystal structure on* $\mathbb{H}(\lambda)$ *determined by these maps corresponds to the crystal structure defined by Definition 29.*

Proof  By Definition 29, Proposition 48, and Proposition 44, the statement follows. ∎

The crystal structure on $\mathbb{H}(\lambda)$ $(\lambda \in P^+)$ is also directly computed by Theorem 42. The following algorithms compute the maps $\varphi_i, \varepsilon_i, f_i, e_i$ $(i \in I)$ based on Theorem 42. Note that the map wt is simply computed by $\sum_{k \in I}(\mu_k - \mu_{k+1})\Lambda_k$ for $H = (\lambda, \mu, 0, (U_{ij})_{i<j})$.

The following is an example of executing Algorithm 8.

**Algorithm 6** Algorithm for $\varphi_i$ on $\mathbb{H}(\lambda)$

**Input:** $H = (\lambda, \mu, 0, (U_{ij})_{i<j}) \in \mathbb{H}(\lambda)$, $i \in I$
**Output:** $\varphi_i(H)$

```
φ_i(H) := 0
for k = 1, 2, ..., i do
    φ_i(H) := max(U_{ki} − U_{k+1,i+1} + φ_i(H), 0)
end for
return φ_i(H)
```

**Algorithm 7** algorithm for $\varepsilon_i$ on $\mathbb{H}(\lambda)$

**Input:** $H = (\lambda, \mu, 0, (U_{ij})_{i<j}) \in \mathbb{H}(\lambda)$, $i \in I$
**Output:** $\varepsilon_i(H)$

```
ε_i(H) := 0
for k = 1, 2, ..., i do
    ε_i(H) := max(U_{i+2−k,i+1} − U_{i+1−k,i} + ε_i(H), 0)
end for
ε_i(H) = max(U_{1,i+1} + ε_i(H), 0)                    ▷ For k = i + 1
return ε_i(H)
```

**Algorithm 8** Algorithm for $f_i$ on $\mathbb{H}(\lambda)$

**Input:** $H = (\lambda, \mu, 0, (U_{ij})_{i<j}) \in \mathbb{H}(\lambda)$, $i \in I$
**Output:** $f_i H$

```
if φ_i(H) = 0 then
    return 0
end if
F := [0]                                               ▷ Set an array
for k = 1, 2, ..., i do
    F := F.append(max(U_{ki} − U_{k+1,i+1} + F[k − 1], 0))
end for
k_{f_i H} := 1
for k = i, i − 1, ..., 1 do
    if F[k] < 0 then
        k_{f_i H} := k − 1
        break
    end if
end for
μ_i := μ_i − 1
μ_{i+1} := μ_{i+1} + 1
U_{k_{f_i},i} := U_{k_{f_i},i} − 1
U_{k_{f_i},i+1} := U_{k_{f_i},i+1} + 1
return (λ, μ, 0, (U_{ij})_{i<j})
```

**Example 51**
*Let* $n = 4$, $\lambda = \mu = \Lambda_1 + \Lambda_3$. *Note that* $\Lambda_1 + \Lambda_3$ *is represented by the partition* $(2, 1, 1, 0)$. *Let* $H = (\lambda, \mu, 0, (0)_{k<l}) \in \mathbb{H}(\lambda)$. *The action of* $f_1$ *on* $\mathbb{H}(\lambda)$ *is computed as follows by Algorithm 8. Let* $i = 1$. *Set* $F = [0]$. *Since* $U_{11} - U_{22} + F[0] = 1$, *set* $F = [0, 1]$. *Set* $k_{f_i H} = 1$. *Since* $F[1] = 1 > 0$, *we have* $k_{f_i H} = 1$. *Then set* $\mu_1 = \mu_1 - 1 = 1$, $\mu_2 = \mu_2 + 1 = 2$, $U_{11} = U_{11} - 1 = 1$,

**Algorithm 9** Algorithm for $e_i$ on $\mathbb{H}(\lambda)$

**Input:** $H = (\lambda, \mu, 0, (U_{ij})_{i<j}) \in \mathbb{H}(\lambda)$, $i \in I$
**Output:** $e_i H$
  **if** $\varepsilon_i(H) = 0$ **then**
    **return** 0
  **end if**
  $E := [0]$
  **for** $k = 1, 2, \ldots, i+1$ **do**
    $E := E.\mathrm{append}(\max(U_{i+2-k,i} - U_{i+1-k,i+1} + E[k-1], 0))$
  **end for**
  $k_{e_i H} := 1$
  **for** $k = i+1, i, \ldots, 1$ **do**
    **if** $E[k] < 0$ **then**
      $k_{e_i H} := k - 1$
      **break**
    **end if**
  **end for**
  $\mu_i := \mu_i + 1$
  $\mu_{i+1} := \mu_{i+1} - 1$
  $U_{k+2-k_{e_i},i} := U_{k+2-k_{e_i},i} + 1$
  $U_{k+2-k_{e_i},i+1} := U_{k+2-k_{e_i},i+1} - 1$
  **return** $(\lambda, \mu, 0, (U_{ij})_{i<j})$

*and* $U_{12} = U_{12} + 1 = 1$. *Then we have* $f_1 H = (\lambda, \mu, 0, (U_{ij})_{i<j})$. *See Fig. 6.*

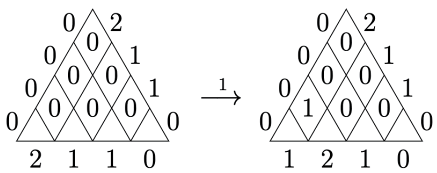



Fig. 6: Action of $f_1$ on the $U_q(\mathfrak{sl}_4)$-crystal $\mathbb{H}(\Lambda_1 + \Lambda_3)$

Algorithms 6, 7, 8, and 9 compute $\varphi_i, \varepsilon_i, f_i, e_i$, $(i \in I)$ according to Theorem 42.

**Proposition 52**
*For* $\lambda \in P^+$, *let* $H \in \mathbb{H}(\lambda)$. *Let* $i \in I$.

1. *Algorithm 6 with inputs* $H$ *and* $i$ *yields* $\varphi_i(H)$.
2. *Algorithm 7 with inputs* $H$ *and* $i$ *yields* $\varepsilon_i(H)$.
3. *Let* $K$ *be the result of Algorithm 8 with inputs* $H$ *and* $i$. *Then,* $K = f_i H$.

4. *Let $K$ be the result of Algorithm 9 with inputs $H$ and $i$. Then, $K = e_i H$.*

Proof (1) and (2) immediately follow from Theorem 42. (3) is proved if $k_{f_i H}$ in Algorithm 8 corresponds to the one in Theorem 42.

For $\lambda \in P^+$, let $H = (\lambda, \mu, 0, (U_{ij})_{i<j}) \in \mathbb{H}(\lambda)$. We can assume $\varphi_i(H) > 0$. This means that $k_{f_i H}$ is defined and

$$\varphi_i(H) = \sum_{k=k_{f_i H}}^{n} (U_{ki} - U_{k+1,i+1}).$$

In particular, $\varphi_i^{(k_{f_i H}-1)}(H) = 0$ and $\varphi_i^{(k_{f_i H})}(H) = U_{k_{f_i H},i} - U_{k_{f_i H}+1,i} > 0$ hold by the definition of $k_{f_i H}$. Then we have

$$\varphi_i^{(m)}(H) = \sum_{k=k_{f_i H}}^{m} (U_{ki} - U_{k+1,i+1}) > 0 \quad (m = k_{f_i H}, k_{f_i H} + 1, \ldots, i).$$

By Theorem 42, $F$ in Algorithm 8 is an array of $\varphi_i^{(l)}(H)$ such that $F[l] = \varphi_i^{(l)}(H)$ for $l \in [i]$. Then $max\{k \in [i] \mid F[k] < 0\} = k_{f_i H} - 1$ holds, hence $k_{f_i H}$ in Algorithm 8 corresponds to the one in Theorem 42. Similarly, (4) can be shown. ∎

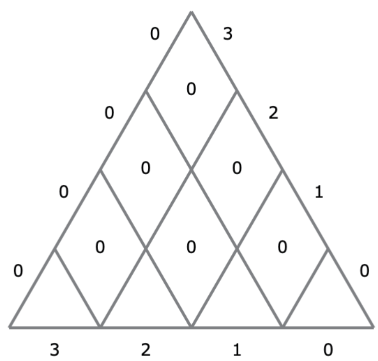
# 4 Examples by *khive-crystal*

In this section, we show some examples of executing the algorithms given in Section 3. These examples are computed by the originally implemented Python package named *khive-crystal* [13]. Then we also give the usage of *khive-crystal*.

In *khive-crystal*, K-hive can be declared by the function *khive*. Furthermore, we can show a K-hive as an image using the function *view*. The following code is an example of functions of *khive* and *view*.

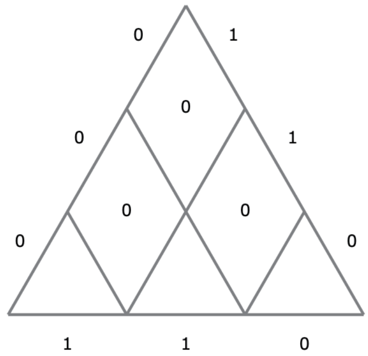
```
» from khive_crystal import khive, view
» H = khive(
..    n=4, alpha=[3, 2, 1, 0], beta=[3, 2, 1, 0], gamma=[0, 0, 0, 0], Uij=[[0, 0, 0], [0, 0], [0]]
.. )
» H
KHive(n=4, alpha=[3, 2, 1, 0], beta=[3, 2, 1, 0], gamma=[0, 0, 0, 0], Uij=[[0, 0, 0], [0, 0], [0]])

» view(H)
```

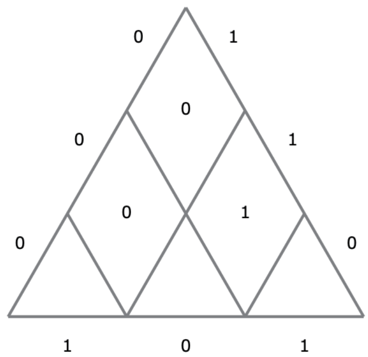


The following codes compute the crystal structure on $U_q(\mathfrak{sl}_3)$-crystal $\mathbb{H}(\Lambda_2)$ by Algorithms 1 and 2.

```
» from khive_crystal import e, epsilon, f, khive, phi, view
» H = khive(n=3, alpha=[1, 1, 0], beta=[1, 1, 0], gamma=[0, 0, 0], Uij=[[0, 0], [0]])
» view(H)
```

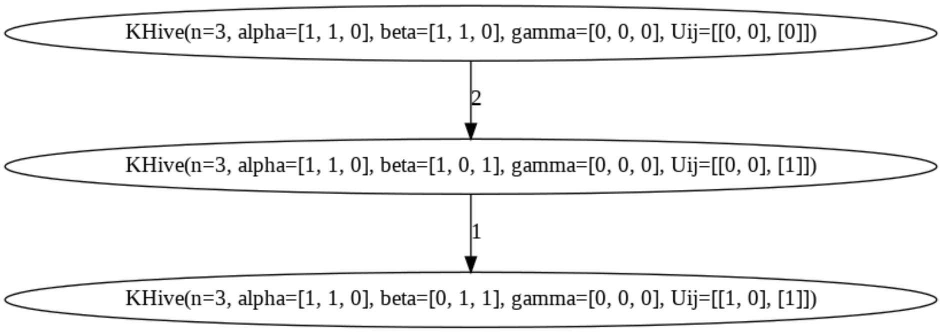



```
» f(i=1)(H)
# None
» view(f(i=2)(H))
```




The crystal graph of $\mathbb{H}(\Lambda_2)$ can be shown by the function called *crystal_graph*, where the function *khives* is the function to declare $\mathbb{H}(\Lambda_2)$.

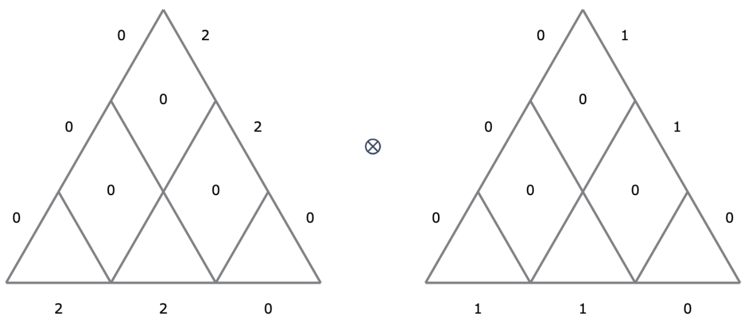
```
» from khive_crystal import khives, crystal_graph
» crystal_graph(khives(n=3, alpha=[1, 1, 0]))
```



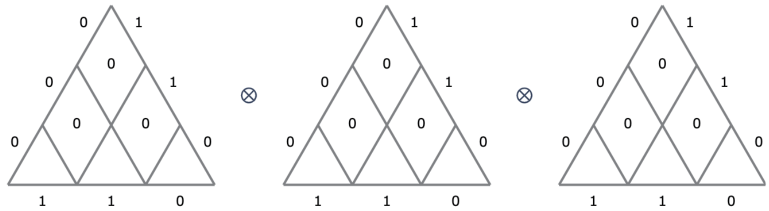

Note that the crystal graph is realized by the open source graph visualization software called *Graphviz*.

The crystal structure on $\mathbb{H}(\lambda)\,(\lambda \in P^+)$ is defined by algorithms of the crystal structure of $\mathbb{H}(\Lambda_k)\,(k \in I)$, $\Psi_\lambda$, $\Psi$, and $\Psi^{-1}$. Then we first show an example for Algorithms 3, 4, and 5, which are implemented as functions *psi_lambda*, *psi*, and *psi_inv*, respectively. The following code is an example for $\Psi_{(3,3,0)}$ and $\Psi$ for $\mathbb{H}((3,3,0))$.

```
» from khive_crystal import khive, psi, psi_lambda, view
» H = khive(n=3, alpha=[3, 3, 0], beta=[3, 3, 0], gamma=[0, 0, 0], Uij=[[0, 0], [0]])
» psi_lambda(H)
[
    KHive(n=3, alpha=[2, 2, 0], beta=[2, 2, 0], gamma=[0, 0, 0], Uij=[[0, 0], [0]]),
    KHive(n=3, alpha=[1, 1, 0], beta=[1, 1, 0], gamma=[0, 0, 0], Uij=[[0, 0], [0]])
]
» view(psi_lambda(H))
```

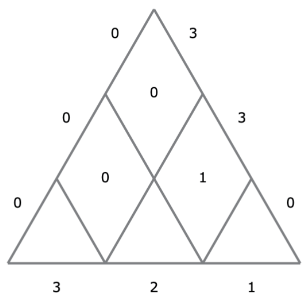


```
» psi(H)
[
    KHive(n=3, alpha=[1, 1, 0], beta=[1, 0, 0], gamma=[0, 0, 0], Uij=[[0, 0], [0]]),
    KHive(n=3, alpha=[1, 1, 0], beta=[1, 0, 0], gamma=[0, 0, 0], Uij=[[0, 0], [0]]),
    KHive(n=3, alpha=[1, 1, 0], beta=[1, 1, 0], gamma=[0, 0, 0], Uij=[[0, 0], [0]])
]
» view(psi(H))
```



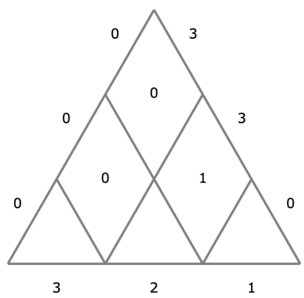

Then we show examples of algorithms of $f_i$ for $\mathbb{H}(\lambda)$. The following code is an example of $f_2$ for $\mathbb{H}((3,3,0))$.

```
» from khive_crystal import khive, psi, psi_inv, view
» H = khive(n=3, alpha=[3, 3, 0], beta=[3, 3, 0], gamma=[0, 0, 0], Uij=[[0, 0], [0]])
» psi_inv(f(i=2)(psi(H))) # = f_i(H)
```




The crystal structure on $\mathbb{H}(\lambda)$ ($\lambda \in P^+$) is also computed by Algorithms 8 and 9.

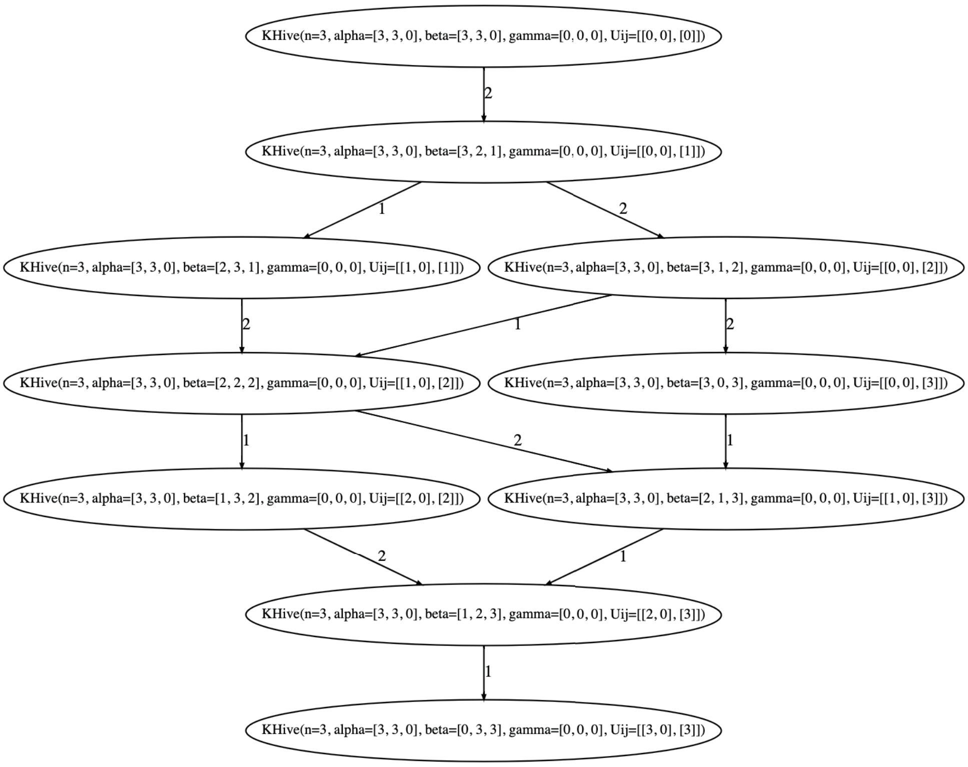
```
» from khive_crystal import khive, e, epsilon, f, phi
» H = khive(n=3, alpha=[3, 3, 0], beta=[3, 1, 0], gamma=[0, 0, 0], Uij=[[0, 0], [0]])
» phi(i=2)(H)
3
» view(f(i=2)(H))
```




The crystal graph of $\mathbb{H}((3,3,0))$ is the following.

```
» from khive_crystal import khives, crystal_graph
```

» crystal_graph(khives(n=3, alpha=[3, 3, 0]))




# 5 Concluding Remarks

In this paper, two approaches are given for a set of algorithms for crystal structures on $\mathbb{H}(\lambda)$ for $\lambda \in P^+$. One approach can be obtained by considering $\mathbb{H}(\lambda)$ as a subset of a tensor product of the form $\mathbb{H}(\Lambda_k)$ with $k \in I$. This method also provides an algorithm to embed a K-hive into the tensor products of K-hives whose right edge labels are determined by a fundamental weight. The other approach can be obtained by considering a combinatorial description of the crystal structure on $\mathbb{H}(\lambda)$.

Recall that $\mathbb{H}(\lambda)$ realizes the crystal basis of the irreducible highest weight module of the highest weight $\lambda$. Then, we can compute the action of $U_q(\mathfrak{sl}_n)$ on $V(\lambda)$ at $q = 0$ and apply it to compute other representation problems by crystals of K-hives. For example, the tensor product decomposition problem may be one of the problems, which can be computed from crystals of K-hives.

# Acknowledgements

We would like to express our gratitude to Professor Itaru Terada of the University of Tokyo for his valuable comments and constructive suggestions. Additionally, we would like to thank the anonymous referees for their helpful suggestions in improving our manuscript.

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