Rational approximants of formal power multivariate series

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1 Introduction

For any formal power multivariate series $s \in K[[X_i]]_{1 \leq i \leq n}$ on a field K, we propose an algorithm for computing a family of rational series $(s_k)_{k \in N}$ such that the difference $s - s_k$ is at least of order k.
The method consists in using the computation in noncommutative variables: for any order k, we construct a rational (recognizable) series $s_{nc_k} \in K\langle\langle X_i \rangle\rangle_{1 \leq i \leq n}$ in noncommutative variables, of minimal rank r_k, such that s_k is its commutative image. This construction uses the Hankel matrix $H(s_{nc_k})$ of the series s_{nc_k}. The rational series s_k is provided as a quotient of 2 polynomials, the denominator being at most of total degree r_k.
If s is rational, there is an order k_0 such that $\forall k \geq k_0$, the rank of s_{nc_k} is r_{k_0}. And then for $k \geq k_0$, the commutative image of s_{nc_k} is $s = P/Q, Q$ being at most of total degree r_{k_0}.

2 Algorithm

We know that a rational series in noncommutative variables

$$s_{nc} = \sum_{w \in X_1, \cdots, X_n *} \langle s_{nc} | w \rangle w$$

is such that its Hankel matrix $H(s_{nc} = (\langle s_{nc} | w_1 \cdot w_2 \rangle)_{w_1, w_2 \in \{X_1, \cdots, X_n\}^* \times \{X_1, \cdots, X_n\}^*}$ has a finite rank([**?**, **?**]). Two presentations of s_{nc} are then availables ([**?**]): by a finite weighted automaton A or by a regular expression E.

The algorithm is the following: For any order k, we construct the Hankel matrix of s_{nc_k} by assigning to $\langle s_{nc_k}|X_i^p\rangle$ the value of $\langle s|X_i^p\rangle$ and by maintaining the linear dependence relations existing in the already built part of the matrix while taking into account the relation

$$\sum_{|w|_{X_1=p_1},\cdots|w|_{X_n=p_n}} \langle s_{nc_k}|w\rangle = \langle s|X_1^{p_1}\cdots X_n^{p_n}\rangle$$

We obtain a noncommutative series s_{nc_k} of minimal rank r_k and then the corresponding regular expression E_k. Its commutative image is $s_k = P_k/Q_k$, the total degree of $Q_k \leq r_k$ and $ord(s - P_k/Q_k) \geq k$.

Example: $s = \sum_{i,j\in N} X_1^i X_2^j (\sum_{k=0}^{inf(i,j)} \begin{pmatrix} |i-j|+2k \\ k \end{pmatrix})$

For $k = 2$, the Hankel matrix $H(s_{nc_2})$ is

	ϵ	X_1	X_2	X_1^2	X_1X_2	X_2X_1	X_2^2	$\cdots$
ϵ	1	1	1	1	2*	1*	1	$\cdots$
X_1	1	1	2*					$\cdots$
X_2	1	1*						$\cdots$
X_1^2	1							$\cdots$
X_1X_2	2*							$\cdots$
X_2X_1	1*							$\cdots$
X_2^2	1							$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

where the values marked by a star are computed according to the algorithm. Then for instance, $s_{nc_2} = (X_1+2X_2X_2^*X_1)(1+X_2X_2^*)$ and $s_2 = \frac{1}{1-(X_1+X_2+X_1X_2)}$. For $k \geq 4$, $s_{nc_k} = s_{nc} = [X_1 + X_2(X_2 + 2X_1X_2)^* X_1(X_1 - X_2)]^*[1 + X_2(X_2 + 2X_1X_2)^*(1 + 2X_1)]$ and

$$s_k = s = \frac{1}{(1 - X_1X_2)(1 - (X_1 + X_2))}$$

3 Conclusion

If s is rational, then there is an order k_0 such that $\forall k \geq k_0, s_k = P/Q = s$. Otherwise, this method provides a family of rational approximants (P_k/Q_k) such that the total degree of Q_k is $\leq r_k$, r_k being the minimal rank of the noncommutative intermediary series s_{nc_k}.

References

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